Bode Plots

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The Bode magnitude and phase plots are tools for analyzing the frequency response of linear circuits or systems that can be defined in terms of transfer functions. To construct the bode plots, the transfer function \( H(s) \), must be given in factored form:

\[
H(s) = \frac{K \cdot \prod_{i=1}^{n_z} (s + \omega z_i)}{\prod_{i=1}^{n_p} (s + \omega p_i)} = \frac{K \cdot (s + \omega z_1)(s + \omega z_2) \cdots (s + \omega z_{n_z})}{(s + \omega p_1)(s + \omega p_2) \cdots (s + \omega p_{n_p})}
\]

where numerator terms are called zeros, denominator terms are called poles, \( \omega p_i \) and \( \omega z_i \) are constants, \( n_z \) is the number of zeros, and \( n_p \) is the number of poles. Bode’s insight was to realize that if we take the log of the magnitude of both sides of the above equation we have:

\[
\log|H(s)| = \log\left|\frac{(s + \omega z_1)(s + \omega z_2) \cdots (s + \omega z_{n_z})}{(s + \omega p_1)(s + \omega p_2) \cdots (s + \omega p_{n_p})}\right|
\]

\[
= \log|s + \omega z_1| + \log|s + \omega z_2| + \cdots + \log|s + \omega z_{n_z}|
\]

\[
- \log|s + \omega p_1| - \log|s + \omega p_1| - \cdots - \log|s + \omega p_{n_p}|
\]

This formulation allows us to deal with the right-hand side as a sum of terms, rather than as a quotient of products of terms. The basic plan is to find the graph of each of the terms on the right-hand side, and then to "add" the graphs together "by eye" (or by machine with Mathcad).

Decibels

Before we go into the details of this procedure, it is necessary to examine the term on the left-hand side of the above equation. What Bode understood was that the quantity on the left-hand side is, in fact, already in standard units. The units to which we refer, were developed by telephone engineers to conveniently represent ratios of power as an aid to analyzing the attenuation of telephone signals as they are propagated down the telephone wire. The Bel (after Alexander Graham Bell) is defined as
Loss in Bels = \log\left(\frac{P_{\text{in}}}{P_{\text{out}}}\right)

Unfortunately the Bel (like the Farad) is too large a unit to be useful, so the deciBel, which is one tenth of a Bel, is the usual unit.

Loss in DeciBels = 10 \cdot \log\left(\frac{P_{\text{in}}}{P_{\text{out}}}\right)

Now in most electronic and control work the transfer function represents not a ratio of powers, but a ratio of voltages or currents. However, since we may write:

\[ P = \frac{V^2}{R} = I^2 \cdot R \]

we have: Loss in dB = \(10 \cdot \log\left[\frac{V_{\text{in}}^2}{R} \cdot \frac{R}{V_{\text{out}}^2}\right]\) = 20 \cdot \log\left(\frac{V_{\text{in}}}{V_{\text{out}}}\right) \quad \text{or}

\[ \text{Gain in dB} = 10 \cdot \log\left[\frac{V_{\text{out}}^2}{R} \cdot \frac{R}{V_{\text{in}}^2}\right]\]

\[ \text{Loss in dB} = 10 \cdot \log\left(\frac{I_{\text{in}}^2 \cdot R}{I_{\text{out}}^2 \cdot R}\right) = 20 \cdot \log\left(\frac{I_{\text{in}}}{I_{\text{out}}}\right) \quad \text{or} \]

\[ \text{Gain in dB} = 10 \cdot \log\left(\frac{I_{\text{out}}^2 \cdot R}{I_{\text{in}}^2 \cdot R}\right) = 20 \cdot \log\left(\frac{I_{\text{out}}}{I_{\text{in}}}\right) \]
Point-by-Point Addition of Functions

To gain some insight into "adding graphs by eye" the reader is encouraged to examine the examples given below. (u(t) is the Heaviside unit step function.)

Recall that given a function \( f(t) \), then the graph of \( f(t-k) \) is simply the graph of \( f(t) \) shifted to the right by \( k \) units.

\[
\begin{align*}
t &:= -1, -0.999 .. 5 \\
u(t) &:= \text{if } (t > 0, 1, 0) \\
f_1(t) &:= t \cdot u(t) \\
f_2(t) &:= -1 \cdot (f_1(t)) \\
g_1(t) &:= t \\
g_2(t) &:= -t \\
f_3(t) &:= u(t - 2) - u(t - 3) \\
f_4(t) &:= f_1(t - 1) - 2 \cdot f_1(t - 2)
\end{align*}
\]

\[
\begin{array}{cc}
\text{u(t)} & \text{u(t-2)} \\
& \\
\text{f_1(t)} & \text{f_1(t-1)} \\
& \\
\text{f_2(t)} & \text{g_1(t)} \\
& \\
\end{array}
\]
Normalization

Before we proceed with the derivation of the magnitude plots of the individual terms of a transfer function, we introduce an algebraic normalization of the transfer function that makes the individual plots simpler to manipulate. The transfer function is usually given in factored form as:

\[
H(s) = \frac{K \prod_{i=1}^{n_z} (s + \omega z_i)}{\prod_{i=1}^{n_p} (s + \omega p_i)}
\]

where the coefficients of \( s \) have been set to one. A more convenient normalization, for our purposes, is to set the constant coefficients to one as shown below:

\[
H(s) = \frac{K \prod_{i=1}^{n_z} (s + \omega z_i)}{\prod_{i=1}^{n_p} (s + \omega p_i)} = \frac{K \prod_{i=1}^{n_z} (s + \omega z_i)}{\prod_{i=1}^{n_p} (s + \omega p_i)} \cdot \frac{\prod_{i=1}^{n_z} \left( \frac{s}{\omega z_i} + 1 \right)}{\prod_{i=1}^{n_p} \left( \frac{s}{\omega p_i} + 1 \right)}
\]

so

\[
H(s) = \frac{K \prod_{i=1}^{n_z} (s + \omega z_i)}{\prod_{i=1}^{n_p} (s + \omega p_i)} \cdot \frac{\prod_{i=1}^{n_z} \left( \frac{s}{\omega z_i} + 1 \right)}{\prod_{i=1}^{n_p} \left( \frac{s}{\omega p_i} + 1 \right)}
\]
We now wish to find the magnitude plot for each of the terms of \( H(s) \), which we shall add together to form the magnitude plot for the transfer function as a whole. The magnitude plot is a graph of the magnitude of the transfer function (in dB) vs. \( \log(\omega) \). The transfer functions shown above contain three distinct kinds of terms: constant terms \( \frac{s}{\omega z_n} \), zeros \( s + 1 \), and poles \( s \left( \frac{1}{\omega p_n} + 1 \right) \).

**Magnitude Plots of Constant Terms**

Since the magnitude of a transfer function \( H(s) = K \) is not a function of \( \omega \), such transfer functions have magnitude plots as shown below:

\[
K_1 := \frac{2}{7} \quad \omega := 0.1, 0.2.. 10 \quad K_2 := 15
\]

\[
20 \cdot \log(K_1) = -10.881 \quad 20 \cdot \log(K_2) = 23.522
\]
To begin our study of individual terms of the general transfer function $H(s)$, let us use as an example a transfer function that has only one term. Consider the frequency response of the circuit shown below. (This experiment is performed in virtually every undergraduate circuits laboratory.) The transfer function of this circuit is:

$$H(s) = \frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} = \frac{1}{s} \left( \frac{1}{s + 1} \right) = \frac{1}{s + 1}$$
Since we have agreed to deal with magnitudes in dB (in order to decouple the terms of the transfer function from one another), we have plotted the magnitude of the response in dB versus frequency by point-by-point calculation below.

\[ \omega := 0.01, 0.02 \ldots 100 \quad H(\omega) := 20 \cdot \log \left( \frac{1}{1 + \omega \cdot i} \right) \]

We see that for large frequencies \((\omega > 50)\) the plot is nearly linear, but for small frequencies the plot is strongly nonlinear. In fact the plot appears nearly exponential. Since exponential functions have linear graphs when plotted against a logarithmic frequency axis, we plot \(|H(s)|\) versus \(\log(\omega)\) below. (We obtain the semi-log plot by using Format X Y Axes, and changing the x axis to a log scale.)
We see that this plot is nearly linear over most of the frequency range. Our basic strategy is to find equations for the two different straight lines that describe the magnitude response and to extend the straight lines until they intersect as shown below.

\[
\omega_0 := 1 \quad H_{SL}(\omega) := \text{if}(\omega > \omega_0, -20 \cdot \log(\omega), 0)
\]

(We will derive this equation for \( H_{SL} \) presently.)

Consider a more general transfer function of the form:

\[
H(s) = \frac{1}{s \frac{\omega_0}{\omega} + 1}
\]

where \( \omega_0 \) is a constant which we shall call the break frequency for reasons that we shall make clear presently. \( \omega \) without subscript is the frequency variable over which we desire the frequency response.

The magnitude of this transfer function is:

\[
|H(j\omega)|_{dB} = 20 \cdot \log \left( \frac{1}{|j \cdot \frac{\omega}{\omega_0} + 1|} \right)
\]

We seek a straight-line approximation to this magnitude for our magnitude plot. We proceed by finding the limit of the magnitude function as \( \omega \) approaches zero, and using that limit as an approximation of the magnitude function for all "low" frequencies. We then find the limit of the magnitude function as \( \omega \) approaches positive infinity and use that limit as an approximation of the magnitude function for all "high" frequencies. We shall define what we mean by "low" frequencies and "high" frequencies presently.
\[
\lim_{\omega \to 0} 20 \cdot \log \left( \frac{1}{\left| \frac{j \omega}{\omega_0} + 1 \right|} \right) = 20 \cdot \log(1) - 20 \cdot \log(1) = 0 \text{ dB}
\]

Since the limit of the magnitude function as \( \omega \) approaches zero is 0 dB, we approximate the magnitude function as 0 dB for all "low" frequencies. That is we approximate the magnitude function as a horizontal straight line at 0 dB for all "low" frequencies.

For "high" frequencies we take the limit of \( H(j\omega) \) as \( \omega \) approaches positive infinity.

\[
\lim_{\omega \to \infty} 20 \cdot \log \left( \frac{1}{\left| \frac{j \omega}{\omega_0} + 1 \right|} \right) = \lim_{\omega \to \infty} 20 \cdot \log \left( \frac{1}{\left| \frac{j \omega}{\omega_0} \right|} \right)
\]

\[
= \lim_{\omega \to \infty} 20 \cdot \log \left( \left| \frac{\omega_0}{j \omega} \right| \right) = \lim_{\omega \to \infty} \left( 20 \cdot \log(\omega_0) - 20 \cdot \log(\omega) \right) = -20 \cdot \log(\omega)
\]

This is the magnitude approximation at high frequencies.

Decades

To understand the significance of this last function, consider the function \( f(t) = 5t \). This is the equation of a straight line in \( t \), with a slope of 5, passing through the point \( t = 0, f(t) = 0 \).

Similarly the function \( f(\log(\omega)) = -20 \log(\omega) \) is a straight line in \( \log(\omega) \) with a slope of -20 dB passing through 0 dB at \( \omega = 1 \).

\((-20 \log(1) = 0)\) The question is a slope of -20 dB per what? Not -20 dB per radian since we are plotting against \( \log(\omega) \) not against \( \omega \). To answer our question of the units for the slope of our straight line, we must calculate two frequencies \( \omega_1 \) and \( \omega_2 \), such that \( \log(\omega_2) - \log(\omega_1) = 1 \).

Since the slope is the "rise" over the "run", and we desire a slope of \( \frac{-20}{1} \) we need a "rise" of -20 and a "run" of 1. So we have:
\[ \log(\omega_2) - \log(\omega_1) = \log\left(\frac{\omega_2}{\omega_1}\right) = 1 \]

If we exponentiate (base 10) both sides of the right hand equation, we obtain:

\[ 10^{\log\left(\frac{\omega_2}{\omega_1}\right)} = 10^1 \quad \text{or} \quad \frac{\omega_2}{\omega_1} = 10 \]

Two frequencies related in this way are said to be one decade apart. The slope of our "high" frequency approximation is then -20 dB per decade. Of course simply specifying the slope does not uniquely determine a straight line; there are infinitely many straight lines with slope -20 dB per decade. Recalling our earlier analogy with straight lines \( f(t) \), we recognize that \( f(t)=at \) is not the most general equation of a straight line in \( t \): it is the equation of the specific straight line in \( t \) passing through \( (0,0) \). The most general equation of a straight line in \( t \) is \( f(t) = at + b \). To uniquely specify our straight line we must specify one point on the line. The question of which point to specify, and the question of what constitutes a "low" or "high" frequency are considered together. Since the low frequency approximation holds when \( \omega \) approaches zero, i.e. when

\[ \frac{\omega}{\omega_o} \ll 1 \]

We approximate the magnitude function by its low frequency approximation for all \( \omega \) such that

\[ \frac{\omega}{\omega_o} < 1 \quad \text{(i.e. for all } \omega \text{ such that } \omega < \omega_o) \]

That is, we use the low frequency approximation (asymptote) for all \( \omega \) such that \( \omega < \omega_o \). Similarly we use the high frequency approximation (asymptote) for all \( \omega \) such that

\[ \frac{\omega}{\omega_o} > 1 \quad \text{(i.e. for all } \omega \text{ such that } \omega > \omega_o) \]

The question of what point on the -20 dB per decade line to specify is settled by the above approximation strategy: we select the point \( \omega = \omega_o \). The magnitude of a transfer function of the form:

\[ H(s) = \frac{1}{\frac{s}{\omega_o} + 1} \]
is approximated by a straight line of 0 dB for all $\omega < \omega_o$ and a straight line of slope -20 dB per decade passing through the point having magnitude 0 dB at $\omega = \omega_o$, for all $\omega > \omega_o$. That is, $|H(j\omega)|$ is approximated as 0 dB for $\omega < \omega_o$, and $|H(j\omega)|$ is approximated by $20 \log(\omega) - 20 \log(\omega_o)$ for $\omega > \omega_o$.

**e.g.**

\[
H_1(s) := \frac{1}{\frac{s}{0.5} + 1} \quad H_2(s) := \frac{1}{\frac{s}{1} + 1} \quad H_3(s) := \frac{1}{\frac{s}{3} + 1}
\]

\[
H_4(s) := \frac{1}{\frac{s}{12} + 1} \quad \omega := 0.1, 0.2..100
\]

\[
H_1(\omega) := \begin{cases} 
\omega > 0.5, -20 \cdot \log\left(\frac{\omega}{0.5}\right), 0 
\end{cases} \quad H_2(\omega) := \begin{cases} 
\omega > 1, -20 \cdot \log\left(\frac{\omega}{1}\right), 0 
\end{cases}
\]

\[
H_3(\omega) := \begin{cases} 
\omega > 3, -20 \cdot \log\left(\frac{\omega}{3}\right), 0 
\end{cases} \quad H_4(\omega) := \begin{cases} 
\omega > 12, -20 \cdot \log\left(\frac{\omega}{12}\right), 0 
\end{cases}
\]

(We shall discuss these Mathcad functions for plotting straight-line magnitude approximations later when we do the first "First Order Magnitude Example.")
These plots are constructed on semi-log paper by hand as follows. First locate the break frequency $\omega_o$, then draw a straight line of slope -20 dB per decade (downward and to the right) starting at $\omega_o$. In some cases, for instance when $\omega_o = 12$ as above, the frequency that is one decade above $\omega_o$ is not on the graph. In these cases it is helpful if one knows the equivalent of -20 dB per decade in dB per octave.

**Octaves**

Frequencies $\omega_1$ and $\omega_2$ are said to be one octave apart if $\omega_2 = 2 \omega_1$. (This terminology results from musical theory where notes that are separated by eight diatonic pitches are said to be one octave apart. For example middle C and the C one reaches by playing C D E F G A B C, are an octave apart. Two notes that are an octave apart have a frequency ratio of 2:1.) To find the slope in dB per octave that is equivalent to -20 dB per decade we need to know how many decades there are in an octave. For $\omega_1$ and $\omega_2$ to be $x$ decades apart we require:

$$\log\left(\frac{\omega_2}{\omega_1}\right) = x$$

We can check the above result for pairs of frequencies whose separation in decades is known, for example:

<table>
<thead>
<tr>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
<th>Separation in decades</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>100</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
<td>1000</td>
<td>2</td>
</tr>
<tr>
<td>10</td>
<td>10000</td>
<td>3</td>
</tr>
</tbody>
</table>

Now if $\omega_2 = 2 \omega_1$, then

$$\log\left(\frac{\omega_2}{\omega_1}\right) = \log\left(\frac{2 \cdot \omega_1}{\omega_1}\right) = \log(2) = 0.30103$$
So frequencies $\omega_1$ and $\omega_2$ that are one octave apart are 0.30103 decades apart, and
1 octave = 0.30103 decades, and 1 decade = $\frac{1}{0.30103}$ octaves.

Then

$$-20 \cdot \frac{\text{dB}}{\text{decade}} = -20 \frac{\text{dB}}{\text{decade}} \cdot 0.30103 \frac{\text{decade}}{\text{octave}} = -6.0206 \cdot \frac{\text{dB}}{\text{octave}}$$

or, -20 dB per decade is approximately -6 dB per octave. Referring again to the example

$$H_4(s) = \frac{1}{s + 1}$$

while the decade related frequency $\omega = 120$ is not on the graph, the octave related frequency $\omega = 24$ is on the graph. So in this case one would draw a straight line of slope -6 dB per octave starting at $\omega = 12$.

**Magnitude Plots of First order Zeros**

The situation for zeros is very similar. Consider a transfer function $H(s)$ such that

$$H(s) = \left(\frac{s}{\omega_0} + 1\right)$$

The low frequency approximation ($\omega < \omega_0$) is:

$$\left|H(j\omega)\right| \text{ dB} = \lim_{\omega \to 0} 20 \cdot \log \left(\left|\frac{j \cdot \omega}{\omega_0} + 1\right|\right) = \lim_{\omega \to 0} 20 \cdot \log(1) = 0 \text{ dB}$$

The high frequency approximation is:

$$\left|H(j\omega)\right| \text{ dB} = \lim_{\omega \to \infty} 20 \cdot \log \left(\left|\frac{j \cdot \omega}{\omega_0} + 1\right|\right) = \lim_{\omega \to \infty} 20 \cdot \log \left[\left|\frac{j \cdot \omega}{\omega_0}\right|\right]$$

$$= \lim_{\omega \to \infty} 20 \cdot \log \left|\frac{\omega}{\omega_0}\right| = \lim_{\omega \to \infty} \left(20 \cdot \log\left(\omega_0\right) - 20 \cdot \log(\omega)\right)$$
\[
\left| H(j\omega) \right|_{\text{dB}} = 20 \cdot \log(\omega) \cdot \text{dB per decade}
\]

First order zeros with break frequency \(\omega_0\), are handled in a manner similar to the manner in which we handled first order poles with break frequency \(\omega_0\). The magnitude plot is a horizontal straight line through 0 dB for \(\omega < \omega_0\), and a straight line of slope +20 dB per decade (6 dB per octave) through \(\omega_0\), for all \(\omega > \omega_0\).

e.g. 

\[
H_5(s) = \frac{s}{0.5} + 1 \quad H_5(\omega) := \begin{cases} 
\text{if } \omega > 0.5, & 20 \cdot \log\left(\frac{\omega}{0.5}\right), 0 \\
\end{cases}
\]

\[
H_6(s) = s + 1 \quad H_6(\omega) := \begin{cases} 
\text{if } \omega > 1, & 20 \cdot \log\left(\frac{\omega}{1}\right), 0 \\
\end{cases}
\]

\[
H_7(s) = \frac{s}{3} + 1 \quad H_7(\omega) := \begin{cases} 
\text{if } \omega > 3, & 20 \cdot \log\left(\frac{\omega}{3}\right), 0 \\
\end{cases}
\]

\[
H_8(s) = \frac{s}{12} + 1 \quad H_8(\omega) := \begin{cases} 
\text{if } \omega > 12, & 20 \cdot \log\left(\frac{\omega}{12}\right), 0 \\
\end{cases}
\]
Magnitude Plots of First Order Zeros and Poles at $s = 0$

First order magnitude plots for transfer functions that include zeros or poles at $s = 0$, that is for terms where $\omega_0 = 0$, require special attention. Consider a transfer function having a zero at 0, $H(s) = s$. The magnitude function $H(\omega)$ is $H(\omega) = |20 \cdot \log(\omega)|$ for all values of $\omega$. The magnitude plot is a straight line of slope $20$ dB per decade ($6$ dB per octave), passing through $0$ dB at the value of $\omega$ for which $20 \log(\omega) = 0$. Since $\log(1) = 0$, the magnitude function is a straight line of slope $20$ dB per decade ($6$ dB per octave) passing through $\omega = 1$. While this function is relatively easy to draw, it does complicate the process of adding the individual graphs together since there is no frequency range where the slope of the function is zero: the function contributes a slope of $20$ dB per decade at ALL frequencies. Similarly transfer functions containing poles at zero, i.e.,

$$H(s) = \frac{1}{s}$$

have magnitude plots that are straight lines of slope $-20$ dB per decade ($6$ dB per octave) passing through $\omega = 1$. e.g.

$$H_9(s) = s \quad H_9(\omega) := 20 \cdot \log(\omega)$$

$$H_{10}(s) = \frac{1}{s} \quad H_{10}(\omega) := -20 \cdot \log(\omega)$$
Multiple poles and zeros

Transfer functions having terms of the form \( \left( \frac{s}{\omega_0} + 1 \right)^n \) or \( \frac{1}{\left( \frac{s}{\omega_0} + 1 \right)^n} \) are handled exactly as above except that the slope of the magnitude plot of the term having multiplicity \( n \) is not 20 dB per decade (6 dB per octave), but 20 \( n \) dB per decade (6 \( n \) dB per octave). This is true for both zeros and poles except of course that poles have negative slopes. The slope multiplication is also correct when \( \omega_0 = 0 \), i.e. for poles and zeros at the origin (\( s = 0 \)).

First Order Magnitude Examples

#1. As an example of the technique, consider the transfer function:

\[
H(s) = 5 \left[ \frac{s \cdot (s + 3)}{(s + 2) \cdot (s + 4)^2} \right]
\]

First we normalize the transfer function as follows:

\[
H(s) = \frac{5 \cdot 3}{2 \cdot 4^2} \left[ \frac{s \cdot \left( \frac{s}{3} + 1 \right)}{\left( \frac{s}{2} + 1 \right) \cdot \left( \frac{s}{4} + 1 \right)^2} \right] = \frac{15}{32} \left[ \frac{s \cdot \left( \frac{s}{3} + 1 \right)}{\left( \frac{s}{2} + 1 \right) \cdot \left( \frac{s}{4} + 1 \right)^2} \right]
\]

This example has several features that make it useful as an easy first example.

1. There are no break-points \( \omega_0 \) with magnitudes less than 1. (Control System Transfer Functions commonly contain break-points smaller than \( \omega_0 = 1 \).)
2. None of the break points is more than 1 or 2 decades away from \( \omega_0 = 1 \). (Electronic Filter Transfer Functions commonly contain break-points 4, 5, 6 or even 7 decades above \( \omega_0 = 1 \).)

The transfer function has 5 terms: a constant term of \( \frac{15}{32} \), a zero at zero, a zero at 3, a pole at 2 and a pole of multiplicity 2 at 4. The individual term magnitude plots along with the composite plot formed by the addition of the individual terms are given below.
Mathcad Functions for Magnitude Approximations

Consider the task of writing the equation of the linear function graphed below.

\[
x := -10, -9 \times .9 \ldots 10 \quad f(x) := 2 \cdot x
\]

This is very easy, since the y intercept is zero we may write \( f(x) = m \cdot x \) where \( m \) is the easily calculated slope.

Contrast the previous equation with the equation of the linear function graphed below:

\[
g(x) := 2 \cdot x - 5
\]

This is only slightly more difficult since we must find the y intercept \( b \) as well as the slope \( m \), and the function is \( g(x) = m \cdot x + b \). However, notice that when we plot \( |H(j \omega)| \) vs \( \log(\omega) \), we are frustrated since there is no place on the plot where \( \log(\omega) = 0 \). (However many decades smaller than \( \log(\omega) = 1.0 \) we go.)

The key to solving our problem is to note that the second function \( g(x) \) is merely the first function \( f(x) \) shifted to the right by 2.5 units. By substitution of \( x \rightarrow (x - 2.5) \) in \( f(x) \) we have \( g(x) = 2 \cdot (x - 2.5) = 2 \cdot x - 5 \). We are then able to write the equation for \( g(x) \) without knowing the y intercept.
First consider $|H(j\omega)|$ vs log$(\omega)$, where $H(s) = \frac{1}{s}$.

(Note that in Mathcad we Format X Y Axes to set the x axis to a log axis)

Now consider what happens when we replace $[\log(\omega)]$ in $H_{10}(\omega)$ by $[\log(\omega) - \log(3)]$.

Since $H_{10}(\omega) = -20 \cdot \log(\omega)$, when we replace $[\log(\omega)]$ by $[\log(\omega) - \log(3)]$, we get

$$-20 \cdot \left(\log\omega - \log(3)\right) = -20 \cdot \log\left(\frac{\omega}{3}\right)$$

which is $|H(j\omega)|$ vs log$(\omega)$, where $H(s) = \frac{1}{s}$ shifted to the right by 2. (3 - 1 = 2)

Finally consider $|H(j\omega)|$ vs log$(\omega)$, where

$$H(s) = \frac{1}{\frac{s}{3} + 1}$$

Note that $H_3(\omega)$ is 0 for $\omega < 3$ and $H_{10}\left(\frac{\omega}{3}\right)$ for $\omega > 3$. 
The if statement in Mathcad is perfectly suited to generating piece-wise linear functions. The syntax is if(condition, true, false), where condition is a logical (boolean) expression, true is the expression to be returned if the condition is true, and false is the expression to be returned if the condition is false. For example, the magnitude approximation of

\[
H(s) = \frac{1}{s^3 + 1}
\]
is

\[
H(\omega) = \text{if} \left( \omega > 3, -20 \cdot \log \left( \frac{\omega}{3} \right), 0 \right)
\]

In general the magnitude approximation in Mathcad for a term of the form

\[
H(s) = \frac{1}{\omega_0^s + 1}
\]
is

\[
H(\omega) = \text{if} \left( \omega > \omega_0, -20 \cdot \log \left( \frac{\omega}{\omega_0} \right), 0 \right)
\]

The magnitude approximation for a term of the form

\[
H(s) = \frac{s}{\omega_0^s + 1}
\]
is

\[
\text{if} \left( \omega > \omega_0, 20 \cdot \log \left( \frac{\omega}{\omega_0} \right), 0 \right)
\]

For our example then, we have:

\[
M_1(\omega) := 20 \cdot \log \left( \frac{15}{32} \right) \quad M_2(\omega) := \text{if} \left( \omega > 3, 20 \cdot \log \left( \frac{\omega}{3} \right), 0 \right)
\]

\[
M_3(\omega) := 20 \cdot \log(\omega) \quad (\text{See plot of } H_9 \text{ above.})
\]

\[
M_4(\omega) := \text{if} \left( \omega > 2, -20 \cdot \log \left( \frac{\omega}{2} \right), 0 \right) \quad M_5(\omega) := \text{if} \left( \omega > 4, -40 \cdot \log \left( \frac{\omega}{4} \right), 0 \right)
\]
In Mathcad all we have to do to add these functions point-by-point is to define $M_T(\omega)$ as

$$M_T(\omega) := M_1(\omega) + M_2(\omega) + M_3(\omega) + M_4(\omega) + M_5(\omega)$$

and plot $M_T(\omega)$

This is our composite Bode magnitude plot!

To proceed by hand with semi-log paper, we must first discuss some peculiarities of semi-log paper.
Several remarks are in order concerning the use of semi-log paper. First, since the vertical scale is linear, all magnitudes must be converted into dB. For example, the magnitude of the constant term \( \frac{15}{32} \) must be converted to dB as \( 20 \cdot \log \left( \frac{15}{32} \right) = -6.58 \text{ dB} \). Second, since the frequency scale is a log scale it is never necessary to take the log of any frequency: by locating frequencies according to their values as described by the log scale, the curve is automatically scaled horizontally according to \( \log(\omega) \). Finally the frequency scale must be oriented so that the distance between successive numbers decreases as one proceeds from left to right; this usually means placing the log scale markings on top. The log scale runs from 1 to 1 three times on 3 cycle semi-log paper. The scale markings must be identified as to their magnitudes as shown above in our composite Bode magnitude plot. Using the scale in our example #1 above, e.g., the first cycle is 0.1, 0.2, 0.3, ... 0.9, 1.0, the second cycle is 1.0, 2.0, 3.0, ... 9.0, 10.0, the third cycle is 10.0, 20.0, 30.0, 90.0, 100.0. The frequency scale can be set to span any three decade frequency range that is desired. We could have for example, 0.01 to 10.0, or 1 to 1000.0, or 1000 to 1000000, or any range that is necessary to represent the magnitude plot of the transfer function at hand. To follow the steps we now give, refer to the figure that follows the steps.

**The composite drawing is produced on semi-log paper as follows:**

1) Add all the contributions to the magnitude at \( \omega = 1 \). In the case of our present example this would be:
\[
-6.58 + 0 + 0 + 0 + 0 = -6.58 \text{ dB}.
\]
Enter point #1.

2) Determine the behavior of the composite function for frequencies smaller that \( \omega = 1 \). In our example the only contribution to the magnitude function for \( \omega < 1 \) is the contribution from the zero at zero (the \( s \) term in the numerator). Therefore at \( \omega = 0.1 \) the magnitude is 20 dB smaller than it is at \( \omega = 1 \) since the slope of the magnitude function is +20 dB per decade. So the Magnitude at \( \omega = 0.1 = -6.58 - 20 = -26.58 \). Enter point #2

3) Determine the behavior of the composite function for \( \omega > 1 \). Again in our example, for \( 1 < \omega < 2 \), the only contribution to the magnitude function is from the zero at zero, so extend the +20 dB per decade line to \( \omega = 2 \) and enter point #3.

\[
(-6.58 \text{ dB} + \left( \log \left( \frac{2}{1} \right) \cdot \text{decades} \cdot \frac{20 \cdot \text{dB}}{\text{decade}} \right) = -0.56 \text{ dB})
\]
4) For \(2 < \omega < 3\), we have two contributions to the composite magnitude function: the contribution of the zero at zero, and the contribution of the pole at 2. Since the zero at zero contributes a slope of +20 dB per decade and the pole at 2 contributes a slope of -20 dB per decade, the contributions cancel each other out and the composite magnitude function does not change between \(\omega = 1\) and \(\omega = 2\), so draw a horizontal line from point #3 to \(\omega = 3\). Enter point #4.

5) For \(\omega > 3\) we must add the contribution of the zero at 3. This adds another slope of +20 dB per decade so we would like to draw a line with slope +20 dB per decade between \(\omega = 3\) and \(\omega = 30\). Unfortunately the magnitude 19.5 dB is beyond the range of our graph, so we draw a line with slope +6 dB per octave between \(\omega = 3\) and \(\omega = 6\), giving us point #5. 

\[
(-0.56 + \log\left(\frac{6}{3}\right)) \cdot 20 = 5.46\text{dB}
\]

6) At \(\omega = 4\), the composite function changes character again as we must add the contribution of the double pole (pole of multiplicity 2) at 4. Enter point #6 (Point #6 is where the line we drew in step 5 intersects \(\omega = 4\))

\[
(-0.56 + \log\left(\frac{4}{3}\right)) \cdot 20 = 1.94\text{dB}
\]

7) For \(\omega > 4\) the double pole at \(\omega = 4\) adds a slope of -40 dB per decade so the total slope for \(\omega > 4\) is +20 - 20 + 20 - 40 = -20 dB per decade, so enter point #7.

\[
1.94 - 20 = -18.06
\]

This completes the straight-line approximation to the magnitude of the transfer function

\[
H(s) = 5 \cdot \frac{s \cdot (s + 3)}{(s + 2) \cdot (s + 4)^2}
\]

points :=

\[
\begin{array}{ll}
0.1 & -26.58 \\
1.0 & -6.58 \\
2.0 & -0.56 \\
3.0 & -0.56 \\
4.0 & 1.94 \\
6.0 & 5.46 \\
40 & -18.06
\end{array}
\]

\(L(\omega)\) is the construction line in step 5.

\[
L(\omega) := -0.56 + 20 \cdot \log\left(\frac{\omega}{3}\right)
\]
Example #2

\[ H(s) = \frac{(s + \frac{1}{4})(s + 5)}{s \left( s + \frac{1}{2} \right)} = \frac{\frac{1}{4} \cdot 5 \left[ \frac{s}{\frac{1}{4}} + 1 \right] \cdot \left( \frac{s}{5} + 1 \right)}{\frac{1}{2} \cdot s \left[ \frac{s}{\frac{1}{2}} + 1 \right]} \]

\[ = \frac{5}{2} \cdot \frac{1}{4} \left[ \frac{s}{\frac{1}{4}} + 1 \right] \cdot \left( \frac{s}{5} + 1 \right) \cdot \left[ \frac{s}{\frac{1}{2}} + 1 \right] \]

\[ \omega := 0.1, 0.15 \ldots 10 \]
\[
M_1(\omega) := 20 \cdot \log \left( \frac{5}{2} \right) \quad \text{and} \quad 20 \cdot \log \left( \frac{5}{2} \right) = 7.959
\]

\[
M_2(\omega) := \begin{cases} \omega > \frac{1}{4}, & 20 \cdot \log \left( \frac{\omega}{1/4} \right), 0 \\ \omega \leq \frac{1}{4}, & 0 \end{cases}
\]

\[
M_3(\omega) := \begin{cases} \omega > \frac{5}{2}, & 20 \cdot \log \left( \frac{\omega}{5} \right), 0 \\ \omega \leq \frac{5}{2}, & 0 \end{cases}
\]

\[
M_4(\omega) := -20 \cdot \log(\omega)
\]

\[
M_5(\omega) := \begin{cases} \omega > \frac{1}{2}, & -20 \cdot \log \left( \frac{\omega}{1/2} \right), 0 \\ \omega \leq \frac{1}{2}, & 0 \end{cases}
\]
\[ M_T(\omega) := M_1(\omega) + M_2(\omega) + M_3(\omega) + M_4(\omega) + M_5(\omega) \]

\[
\begin{pmatrix}
0.1 & 27.96 \\
0.25 & 20 \\
0.5 & 20 \\
1.0 & 13.98 \\
5.0 & 0
\end{pmatrix}
\]

At \( \log(\omega) = \frac{1}{4} \) the contribution from the pole at \( \omega = 0 \) is
\[
20 \cdot \log \left( \frac{1}{\frac{1}{4}} \right) = 12.0412 \text{ dB}
\]

the contribution from the constant is
\[
20 \cdot \log \left( \frac{5}{2} \right) = 7.959 \text{ dB}
\]

So the total contribution at \( \log(\omega) = \frac{1}{4} \) is 20 dB \( \Rightarrow \) point #1. Since the magnitude response is -20 dB per decade for \( \omega < \frac{1}{4} \), we may plot point #2 as \( \omega = 0.1 \) and magnitude
\[
20 + 20 \cdot \log \left( \frac{1}{0.1} \right) = 27.96
\]

For \( \frac{1}{4} < \omega < \frac{1}{2} \) the zero at \( \frac{1}{4} \) cancels the pole at zero, and we draw a horizontal line to point #3 at \( \omega = \frac{1}{2} \). For \( \frac{1}{2} < \omega < 5 \), we have slope -20 dB per decade so we plot point #4 at \( \omega = 1.0 \) and magnitude
\[
10 - 20 \cdot \log \left( \frac{1}{\frac{1}{2}} \right) = 13.98 \text{ dB}
\]

Finally for \( 5 < \omega \), we have
\[-20 + 20 - 20 + 20 = 0 \text{ dB}
\]

hence point #5 at \( \omega = 5 \) and magnitude
\[
13.98 - 20 \cdot \log \left( \frac{5}{1} \right) = 0 \text{ dB}.
\]
Example #3

\[ H(s) = \frac{100000 \cdot s}{(s + 10000) \cdot (s + 50000)} = \frac{100000}{10000 \cdot 50000} \cdot \frac{s}{\left(\frac{s}{10000} + 1\right) \cdot \left(\frac{s}{50000} + 1\right)} \]

\[ \omega := 100, 200 \ldots 1000000 \]

\[ M_1(\omega) := 20 \cdot \log\left(\frac{100000}{5 \cdot 10^8}\right) \]

\[ M_2(\omega) := 20 \cdot \log(\omega) \]

\[ M_3(\omega) := \text{if}\, \omega > 10000, -20 \cdot \log\left(\frac{\omega}{10000}\right), 0 \]

\[ M_4(\omega) := \text{if}\, \omega > 50000, -20 \cdot \log\left(\frac{\omega}{50000}\right), 0 \]
\[ M_T(\omega) := M_1(\omega) + M_2(\omega) + M_3(\omega) + M_4(\omega) \]
At \( \log(\omega) = 100 \) we have \(-73.979\) from the constant term and \(0\)dB + 
\((2 \cdot \text{decades}) \cdot \left(20 \cdot \frac{\text{dB}}{\text{decade}}\right) = 40\) dB so the total magnitude at \( \log(\omega) = 100 \) is 
\(40 - 73.979 = -33.979 \Rightarrow \text{point } #1\). For \(100 < \omega < 10^4\) we have 
\(+20\) dB per decade so point #2 at \(\omega = 10^4\) and magnitude \(-33.979 + 20 \cdot \log\left(\frac{10^4}{100}\right)\) 
\(=6.021\). For \(\omega > 10^4\), the pole at \(10^4\) cancels the zero at 0 and we have a horizontal 
line to point #3. For \(5 \cdot 10^4 < \omega \) we have slope \(-20\) dB per decade, so point #4 at 
\(\omega = 5 \cdot 10^5\) and magnitude \(6.021 - 20 \text{ dB} = -13.979\)
First Order Magnitude Errors

It should be noted that there is an error associated with the straight-line approximation to the first order magnitude function, and that the error is greatest at the break frequency \( \omega_0 \) where the assumptions about "high" and "low" frequencies are the weakest. The highest "low" frequency is \( \omega = \omega_0 \), and the lowest "high" frequency is also \( \omega = \omega_0 \). The magnitude of the error can be found by evaluating \( H(j\omega) \) at \( \omega = \omega_0 \). For pole terms, the error is given by the difference between 0 dB and

\[
20 \cdot \log \left( \frac{1}{j \frac{\omega}{\omega_0} + 1} \right) \text{ evaluated at } \omega = \omega_0, \text{ which is } 20 \cdot \log \left( \frac{1}{\sqrt{1^2 + 1^2}} \right) = -20 \cdot \log(\sqrt{2}) = -3.103 \text{ dB}
\]

So for a pole the magnitude is actually at -3 dB rather than at 0 dB as predicted by the straight-line approximation.

For zero terms the error is given by the difference between 0 dB and

\[
20 \cdot \log \left( j \frac{\omega}{\omega_0} + 1 \right) \text{ evaluated at } \omega = \omega_0, \text{ which is } 20 \cdot \log \left( \sqrt{1^2 + 1^2} \right) = 20 \cdot \log(\sqrt{2}) = 3.103 \text{ dB}
\]

So for a zero the magnitude is actually at +3 dB rather that at 0 dB as predicted by the straight-line approximation.

First Order Phase Approximations

The first order phase response is the difference in phase angle between the input sinusoid and the output sinusoid as a function of frequency for a system characterized by a transfer function of the form:

\[
H(s) = s + \omega_0 \quad (\text{First order zero at } \omega_0)
\]
where $\omega_0$ is referred to as the break frequency for reasons that are apparent from our investigation of the magnitude response, and which plays an important role in the phase response as well.

The phase response as a function of frequency is:

$$\Phi(\omega) = \text{atan} \left( \frac{\omega}{\omega_0} \right)$$

A plot of $\Phi$ vs. $\log(\omega)$ for $\omega_0 = 1$ is shown below:

$$\omega := 0.01, 0.02 \ldots 100$$

Our first approach to a straight-line approximation might be to calculate the slope of the phase function at $\omega = \omega_0$ and draw a line with that slope through the point on the phase function where $\omega = \omega_0$. This is the basis for the approach that Nilsson [1] takes to second order phase approximations. The slope of the phase function at $\omega = \omega_0$ is found by evaluating the derivative of the phase function at $\omega = \omega_0$.

The derivative of $\Phi(\omega)$, is

$$\frac{d}{d\omega} \text{atan} \left( \frac{\omega}{\omega_0} \right) = \frac{1}{\omega_0 \cdot \left(1 + \frac{\omega^2}{\omega_0^2}\right)}$$

Which evaluated at $\omega = \omega_0$ gives: slope $= \frac{1}{2 \cdot \omega_0}$

\[\begin{array}{c}
\text{atan} \left( \frac{\omega}{1} \right) \cdot \frac{360}{2 \cdot \pi} \\
\end{array}\]
Unfortunately this is the slope in radians per radian, not radians per decade. To calculate the slope in radians per decade we must calculate the "rise" over the "run" where the "rise" is in radians and the "run" is in decades. Since we are interested in the slope at the point where \( \omega = \omega_o \), we take as the rise, the limit as \( \Delta \) approaches 0 of:

\[
\left[ (\omega_0 + \Delta) - (\omega_0 - \Delta) \right] \cdot \frac{1}{2 \cdot \omega_o} \quad \text{(The difference in } \omega \text{ is } 2 \Delta)
\]

For the run we use the limit as \( \Delta \) approaches 0, of \( 2\Delta \) expressed in decades. Since one decade is defined as the distance between two frequencies \( \omega_1 \) and \( \omega_2 \), where \( \omega_1 \) and \( \omega_2 \) are related as \( \omega_2 = 10 \omega_1 \), we have:

\[
\Delta \omega \text{ (in decades)} = \log(\omega_2) - \log(\omega_1) = \log\left( \frac{\omega_2}{\omega_1} \right)
\]

which gives \( \log\left( \frac{\omega_2}{\omega_1} \right) = 1 \) decade if \( \omega_2 = 10 \omega_1 \) as required. The run is given then, by the limit as \( \Delta \) approaches 0 of:

\[
\log\left( \frac{\omega_0 + \Delta}{\omega_0 - \Delta} \right)
\]

The slope in radians per decade is given then as:

\[
\lim_{\Delta \omega \to 0} \left[ \left[ (\omega_0 + \Delta) - (\omega_0 - \Delta) \right] \cdot \frac{1}{2 \cdot \omega_o} \right] \cdot \log\left( \frac{\omega_0 + \Delta}{\omega_0 - \Delta} \right) \quad \text{which simplifies to } \frac{1}{2} \cdot \ln(10)
\]

(We must use L'Hopital's rule.)

An interesting aspect of this result is that the slope of the phase function (in radians per decade) evaluated at the break frequency \( \omega_o \) is independent of the value of \( \omega_o \). This is, of course, one of the principle reasons for plotting the phase function against \( \log(\omega) \) rather than against \( \omega \). Phase functions for various values of \( \omega_o \) are plotted below to demonstrate the plausibility of the independence of the slope at \( \omega = \omega_o \) from the value of \( \omega_o \).

\[
\frac{1}{2} \ln(10) \cdot \frac{\text{radians}}{\text{decade}} = \frac{1}{2} \ln(10) \cdot \frac{360}{2 \cdot \pi} \cdot \frac{\text{degrees}}{\text{decade}}
\]
\[
\Phi_1(\omega) := \text{atan}\left(\frac{\omega}{0.1}\right) \cdot \frac{360}{2 \cdot \pi}
\]

\[
\Phi_2(\omega) := \text{atan}\left(\frac{\omega}{0.5}\right) \cdot \frac{360}{2 \cdot \pi}
\]

\[
\Phi_3(\omega) := \text{atan}\left(\frac{\omega}{5}\right) \cdot \frac{360}{2 \cdot \pi}
\]

\[
\Phi_4(\omega) := \text{atan}\left(\frac{\omega}{10}\right) \cdot \frac{360}{2 \cdot \pi}
\]

**First Straight-Line Approximation (Inadequate)**

The slope of the phase function evaluated at the break frequency is

\[
\frac{360 \cdot 1}{2 \cdot \pi} \cdot \ln(10) = 65.964 \text{ Deg/Dec}
\]

If we plot a straight line with this slope through the phase function at the break frequency we obtain the following result. (We have normalized the break frequency \(\omega_0\) to 1)
This approximation suffers from two principal shortcomings. The first disadvantage is that the line \( L_1(w) \) underestimates the range of frequencies over which the phase function changes with \( w \). In other words the error, i.e., the difference between \( \Phi(w) \) and \( \Phi_{\text{approx}}(w) \) at \( w_{lo} \) and \( w_{hi} \) is too large. The second problem is the complexity of the expressions for \( w_{lo} \) and \( w_{hi} \).

Let us devise a new straight-line approximation by applying one of the oldest, most fundamental engineering rules of thumb to the selection of \( w_{lo} \) and \( w_{hi} \). The rule of thumb in question is the notion that if quantity \( a \) is less that \( \frac{1}{10} \) of quantity \( b \), then compared to quantity \( b \), quantity \( a \) may be ignored to a first order approximation.

\[
L_1(\omega) := 45 + 65.964 \cdot \log(\omega) \text{ degrees per decade}
\]

\[
\Phi_{\text{approx}}(\omega) = \begin{cases} 
0 & \text{if } \omega < \omega_{lo} \\
45 + 65.964 \cdot \log(\omega) & \text{if } \omega_{lo} < \omega < \omega_{hi} \\
90 & \text{if } \omega_{hi} < \omega
\end{cases}
\]

where \( \omega_{lo} \) is the frequency where \( L_1(\omega) \) intersects 0 degrees, and \( \omega_{hi} \) is the frequency where \( L_1(\omega) \) intersects 90 degrees. Since \( 45 + 64.964 \cdot \log(\omega_{hi}) = 90 \) we have:

\[
\omega_{hi} = \left( \frac{45}{10 \cdot 64.964} \right) \cdot \omega_o \quad \text{or} \quad \omega_{hi} = 4.928 \cdot \omega_o
\]

Since \( 45 + 64.964 \cdot \log(w) = 0 \) we have:

\[
\omega_{lo} = \left( \frac{45}{10 \cdot 64.964} \right) \cdot \omega_o \quad \text{or} \quad \omega_{lo} = 0.2029 \cdot \omega_o
\]

This approximation suffers from two principal shortcomings. The first disadvantage is that the line \( L_1(\omega) \) underestimates the range of frequencies over which the phase function changes with \( \omega \). In other words the error, i.e., the difference between \( \Phi(\omega) \) and \( \Phi_{\text{approx}}(\omega) \) at \( \omega_{lo} \) and at \( \omega_{hi} \) is too large. The second problem is the complexity of the expressions for \( \omega_{lo} \) and \( \omega_{hi} \). Let us devise a new straight-line approximation by applying one of the oldest, most fundamental engineering rules of thumb to the selection of \( \omega_{lo} \) and \( \omega_{hi} \). The rule of thumb in question is the notion that if quantity \( a \) is less that \( \frac{1}{10} \) of quantity \( b \), then compared to quantity \( b \), quantity \( a \) may be ignored to a first order approximation.
For small frequencies, i.e. for frequencies smaller than $\omega_{lo}$, $\Phi(\omega) \rightarrow 0$, so let us choose for $\omega_{lo}$ not $0.2029 \omega_0$, but $\frac{\omega_0}{10}$. Similarly for large frequencies, i.e. for frequencies larger than $\omega_{hi}$, $\Phi(\omega) \rightarrow 90$, so let us choose not $\omega_{hi} = 4.928 \omega_0$, but $\omega_{hi} = 10 \omega_0$. The straight line resulting from this selection changes value from 0 to 90 over a frequency range of $\frac{\omega_0}{10}$ to $10 \cdot \omega_0$.

Since the distance between $\frac{\omega_0}{10}$ and $\omega_0$ is one decade and the distance between $\frac{\omega_0}{10}$ and $10 \cdot \omega_0$ is one decade, the slope of the line is 90 degrees per 2 decades or 45 degrees per decade. (45 deg per decade = $\log(2) \cdot 45$ deg per octave = 13.56 deg per octave.) The new straight-line approximation as well as the original straight-line approximation, are plotted below.

$$L_2(\omega) := 45 + 45 \cdot \log(\omega)$$

In general for a zero at $\omega_0$ we have $\Phi(\omega) =$

$$\begin{cases} 
\omega < \frac{\omega_0}{10}, & 0, \\
\omega < 10 \cdot \omega_0, & 45 + 45 \cdot \log \left( \frac{\omega}{\omega_0} \right), \\
\omega \geq 10 \cdot \omega_0, & 90 
\end{cases}$$
The line $L_2(\omega)$ is the familiar 45 degree per decade straight-line approximation to which all undergraduate engineering students are introduced. The result is too often simply stated with little or no explanation of the underlying derivation. In fact, well over 95% of the over 100 electrical engineering students informally surveyed thought that the slope of $\tan\left(\frac{\omega}{\omega_0}\right)$ evaluated at $\omega = \omega_0$ has a value of 45 degrees per decade. Before we demonstrate how this same line of reasoning can be applied to the second order phase response, let us calculate the errors of $L_1(\omega)$ and $L_2(\omega)$ at their respective $\omega_{lo}$ and $\omega_{hi}$.

At $\omega_{lo}$ we have:

$$\tan(0.2029) \cdot \frac{360}{2 \cdot \pi} = 11.47$$

So the error at $\omega_{lo}$ is 11.47 degrees for $L_1(\omega)$

and

$$\tan(0.1) \cdot \frac{360}{2 \cdot \pi} = 5.711$$

So the error at $\omega_{lo}$ is 5.711 degrees for $L_2(\omega)$

At $\omega_{hi}$ we have:

$$\tan(4.928) \cdot \frac{360}{2 \cdot \pi} = 78.529$$

So the error at $\omega_{hi}$ is $90 - 78.529 = 11.47$ degrees for $L_1(\omega)$

and

$$\tan(10) \cdot \frac{360}{2 \cdot \pi} = 84.289$$

So the error at $\omega_{hi}$ is $90 - 84.289 = 5.711$ degrees for $L_2(\omega)$

A similar analysis for the related transfer function $H(s) = \frac{1}{s + \omega_0}$ gives the following result:

$$L_3(\omega) := -45 - 65.964 \log(\omega)$$

$$L_4(\omega) := -45 - 45 \log(\omega)$$
In general for a pole at $\omega_0$ we have $\Phi(\omega) =$

\[
\begin{cases}
0 & \text{if } \omega < \frac{\omega_0}{10}, \\
0 & \text{if } \omega < 10 \cdot \omega_0, \\
\approx -45 - 45 \log \left( \frac{\omega}{\omega_0} \right), & \text{for all } \omega
\end{cases}
\]

For both transfer functions we see that the 45 degree per decade approximation better represents the phase function because it distributes the error on both sides of the line, whereas the 66 degree per decade approximation (the one based on the slope of the phase function at the break frequency) puts all of the error on one side of the line. In other words the 45 degree per decade line is a better linear curve fit to the phase function. This results in the smaller errors at $\omega_{lo}$ and $\omega_{hi}$.

**Phase Plots of First Order Poles and Zeros at $s = 0$**

The phase function $\Phi(\omega)$ for a transfer function $H(s) = s$ is found by taking the angle of $H(j\omega)$ as follows:

\[
H(j\omega) = j\omega = 0 + j\omega \quad \text{The angle of such a complex number is}
\]

\[
\Phi(\omega) = \tan(\frac{\text{Re}(H(j\omega))}{\text{Im}(H(j\omega))}) = \tan\left( \frac{\omega}{0} \right) = 90 \text{ Deg for all } \omega
\]

$\theta := 0, \frac{\pi}{10}, \ldots, 2\pi$

$\{f_1(\theta) := \text{if} \left( \theta = \frac{\pi}{2}, 1, 0 \right)\}$
The phase function $\Phi(\omega)$ for a transfer function $H(s) = \frac{1}{s}$ is found by taking the angle of $H(j\omega)$ as follows:

$$H(j\omega) = \frac{1}{j\omega} = 0 - \frac{j}{\omega}$$

The angle of such a complex number is

$$\Phi(\omega) = \tan\left(\frac{\text{Re}(H(j\omega))}{\text{Im}(H(j\omega))}\right) = \tan\left[\left(\frac{-\frac{1}{\omega}}{0}\right)\right] = -90\degree \text{ for all } \omega$$

$$f_2(\theta) := \text{if } \theta = \frac{3\pi}{2}, 1, 0$$
Phase Plots of Constant Terms

Constant terms $K$ produce no phase shift at all if $K > 0$ since the angle of $K + j0 = 0$. 

$$f_3(\theta) := \text{if}(\theta = 0, 1, 0)$$

Constant terms $K$ such that $K < 0$ produce a phase shift of $180^\circ$ for all $\omega$, since the angle of $-K + j0$ is $180^\circ$.

$$f_4(\theta) := \text{if}(\theta = \pi, 1, 0)$$

Plot of $1 + j0 = 1$ at angle 0 deg in the complex plane

Plot of $-1 + j0 = 1$ at angle 180 deg in the complex plane
First Order Phase Approximation Examples

#1. \( H(s) = -5 \cdot \frac{s + 1}{s \cdot (s + 12)} \)

\[ \omega := 0.01, 0.02 \ldots 1000 \]

Since the angle of \(-5 - j0 = 180\) degrees, the constant term contributes a phase of 180 degrees for all \(\omega\). (A positive constant has angle = 0 and so would not produce any phase contribution at all.) The pole at \(s = 0\) contributes a phase of -90 degrees for all \(\omega\), so the constant phase is 180 - 90 = 90 degrees. The zero at \(s = 1\) contributes a phase shift that varies from 0 degrees for \(\omega < 0.1\) to 90 degrees for \(\omega > 10\), varying linearly at +45 degrees per decade for \(0.1 < \omega < 10\). The pole at \(s = 12\) contributes a phase shift that varies from 0 degrees for \(\omega < 1.2\) to -90 degrees for \(\omega > 120\), varying linearly at -45 degrees per decade (-13.56 degrees per octave). We begin by calculating the phase shift for all \(\omega < 0.1\) as 180 - 90 = 90 degrees, fixing point 1. The next step is to make an auxiliary drawing like the one below, which assumes that all of the phase change contributed by each pole and zero occurs suddenly at the break point \(\omega_0\) (the solid red line with vertical segments). Next draw in the 45 degrees per decade lines at the proper locations on the same drawing. (These are the blue dashed lines.) We now can see all the contributing slopes at all frequencies. (The black line at 135 degrees help us associate a blue line with a red vertical transition as follows. The red vertical transition, the black line at 135 degrees and the blue line associated with that red vertical transition all intersect at the same point.) The construction of the phase plot itself begins with a straight line of +45 degrees per decade between \(\omega = 0.1\) and \(\omega = 1.2\) (point 2), fixing point 3. Notice that for 1.2 < \(\omega < 10\) we have a slope of +45 -45 = 0 degrees, so construct a horizontal line from point 3 to point 4. Finally we have a slope of -13.6 degrees per octave (point 5) between \(\omega = 12\) and \(\omega = 120\), so construct a straight line of slope -45 degrees per decade (-13.6 degrees per octave) to point 6.
\[ \Phi_1(\omega) := \begin{cases} \omega < \frac{1}{10} & , 0, \text{if } \omega < 10, 45 + 45 \cdot \log\left(\frac{\omega}{1}\right), 90 \end{cases} \]

\[ \Phi_2(\omega) := -90 \]

\[ \Phi_3(\omega) := 180 \]

\[ \Phi_4(\omega) := \begin{cases} \omega < \frac{12}{10} & , 0, \text{if } \omega < 10 \cdot 12, -45 - 45 \cdot \log\left(\frac{\omega}{12}\right), -90 \end{cases} \]

points := \[
\begin{pmatrix}
0.01 & 90 \\
0.1 & 90 \\
1.2 & 138.563 \\
10  & 138.563 \\
120 & 90 \\
10^3 & 90
\end{pmatrix}
\]

\[ \Phi_T(\omega) := 0 + \Phi_1(\omega) + \Phi_2(\omega) + \Phi_3(\omega) + \Phi_4(\omega) \]
At $w = 0.1$ the contributions to the phase function are the constant -90 degrees from the pole at $w = 0$ and the 180 degress from the (-5) constant term so point #1.

For $w < \frac{1}{10}$, the slope is 0 degrees per decade so point #2.

For $\frac{1}{10} < w < 1.2$ we have slope +45 degrees per decade so point #3.

\[
(90 + 45 \cdot \log \left( \frac{1.2}{0.1} \right) = 138.563
\]

For $1.2 < w < 10$ the slope is $+45 - 45 = 0$ degrees per decade so point #4.

For $10 < w < 120$ the slope is -45 degrees per decade so point #5.

\[
(138.563 - 45 \cdot \log \left( \frac{120}{10} \right) = 90 \text{ degrees})
\]

For $w > 120$ we have slope = 0 so point #6.

$H(s) = \frac{15}{32} \cdot \left[ \frac{s \cdot \left( \frac{s}{3} + 1 \right)}{\left( \frac{s}{2} + 1 \right) \cdot \left( \frac{s}{4} + 1 \right)^2} \right] \quad \omega := 0.1, 0.2 \ldots 100$

$\Phi_1(\omega) := \begin{cases} 
\omega < \frac{3}{10}, & 0, \\
\omega < 3 \cdot 10, & 45 + 45 \cdot \log \left( \frac{\omega}{3} \right), 90
\end{cases}$

$\Phi_2(\omega) := \begin{cases} 
\omega < \frac{2}{10}, & 0, \\
\omega < 2 \cdot 10, & -45 - 45 \cdot \log \left( \frac{\omega}{2} \right), -90
\end{cases}$

$\Phi_3(\omega) := \begin{cases} 
\omega < \frac{4}{10}, & 0, \\
\omega < 4 \cdot 10, & -45 - 45 \cdot \log \left( \frac{\omega}{4} \right), -90
\end{cases}$
\[ A := \begin{pmatrix} 0.1 & 90 \\ 2 & 90 \\ 3 & 0 \\ 3 & 90 \\ 4 & 90 \\ 4 & 0 \\ 100 & 0 \end{pmatrix} \]

\[ E := \begin{pmatrix} 2 & 90 \\ 3 & 0 \\ 4 & 90 \\ 0 & 90 \\ 20 & 0 \\ 30 & 90 \\ 40 & 0 \end{pmatrix} \]

\[ B := \begin{pmatrix} 2 & 90 \\ 10 & 0 \\ 20 & 0 \end{pmatrix} \]

\[ C := \begin{pmatrix} 3 & 0 \\ 10 & 0 \\ 30 & 90 \end{pmatrix} \]

\[ D := \begin{pmatrix} 4 & 90 \\ 10 & 90 \\ 40 & 0 \end{pmatrix} \]
\[\Phi_T(\omega) := 90 + \Phi_1(\omega) + \Phi_2(\omega) + \Phi_3(\omega)\]

\[
\begin{bmatrix}
0.1 & 90 \\
0.2 & 90 \\
0.3 & 82.076 \\
0.4 & 82.076 \\
20 & 5.622 \\
30 & 5.622 \\
40 & 0 \\
100 & 0 \\
\end{bmatrix}
\]

\[
\Phi_T(\omega) = 90 + 0.1 \cdot 50 + 0.2 \cdot 10 + 0.3 \cdot 60 + 0.4 \cdot 10 + 20 \cdot 20 + 30 \cdot 20 + 40 \cdot 20 + 100 \cdot 20
\]

For all \(\omega < \frac{2}{10}\), the slope of \(\Phi_T(\omega)\) is zero, and the only contribution is from the zero at \(\omega = 0\), so point #1 and point #2. For \(\frac{2}{10} < \omega < \frac{3}{10}\), the slope is -45 degrees per decade so point #3. \((90 - 45 \cdot \log \left(\frac{0.3}{0.2}\right) = 82.076\) degrees.\) For \(\frac{3}{10} < \omega < \frac{4}{10}\), the slope is -45 + 45 = 0 so point #4. For \(\frac{4}{10} < \omega < 2 \cdot 10\), the slope is -45 + 45 - 45 = -45 degrees per decade so point #5. \((82.076 - 45 \cdot \log \left(\frac{20}{0.4}\right) = 5.622)\)
For $20 < \omega < 30$, the slope is $+45 - 45 = 0$ degrees per decade so point #6.

For $30 < \omega < 40$, the slope is $-45$ degrees per decade so point #7. $(5.622 - 45 \cdot \log \frac{40}{30}) = 0$) For all $\omega > 40$, the slope is 0 so point #8.

#3.

$$H(s) = \frac{(s + \frac{1}{4})(s + 3)}{s \left(s + \frac{1}{2}\right)} = \frac{1}{4} \cdot 3 \cdot \frac{s + \frac{1}{4} + 1}{s \left(s + \frac{1}{2}\right)} \cdot \frac{s + \frac{1}{3} + 1}{s \left(s + \frac{1}{2}\right)}$$

$$\omega := 0.01, 0.015.. 100$$

$$\Phi_1(\omega) := \begin{cases} \frac{1}{10} & \text{if } \omega < \frac{1}{4}, \\ 0 & \text{if } \omega < 10 \cdot \frac{1}{4}, 45 + 45 \cdot \log \left[\frac{\omega}{\frac{1}{4}}\right], 90 \end{cases}$$

![Graph of \(\Phi_1(\omega)\)](image)
\[
\Phi_2(\omega) := \begin{cases} 
\text{if } (\omega) < \left(\frac{1}{2}\right)_{10}, 0, & \text{if } \omega < 10 \cdot \frac{1}{2}, -45 - 45 \cdot \log\left[\omega \left(\frac{1}{2}\right)\right], -90 
\end{cases}
\]

\[
\Phi_3(\omega) := \begin{cases} 
\text{if } \omega < \frac{3}{10}, 0, & \text{if } \omega < 10 \cdot 3, 45 + 45 \cdot \log\left(\frac{\omega}{3}\right), 90 
\end{cases}
\]

\[
\\
A := \begin{pmatrix} 
0.01 & -90 \\
\frac{1}{4} & -90 \\
\frac{1}{4} & 0 \\
\frac{1}{2} & 0 \\
\frac{1}{2} & -90 \\
3 & -90 \\
3 & 0 \\
100 & 0 \\
\end{pmatrix}
\]

\[
B := \begin{pmatrix} 
\frac{1}{4} & -90 \\
\frac{1}{4} \cdot 10 & 0 \\
\end{pmatrix}
\]

\[
C := \begin{pmatrix} 
\frac{1}{2} & 0 \\
\frac{1}{2} \cdot 10 & -90 \\
\end{pmatrix}
\]

\[
D := \begin{pmatrix} 
\frac{3}{10} & -90 \\
30 & 0 \\
\end{pmatrix}
\]

\[
E := \begin{pmatrix} 
\frac{1}{4} & -90 \\
\frac{1}{2} & 0 \\
\frac{3}{10} & -90 \\
\frac{1}{4} \cdot 10 & 0 \\
\frac{1}{2} \cdot 10 & -90 \\
30 & 0 \\
\end{pmatrix}
\]
\[
\Phi_T(\omega) := -90 + \Phi_1(\omega) + \Phi_2(\omega) + \Phi_3(\omega)
\]
For \( \omega < 0.01 \), the only contribution is from the pole at zero so point #1 and point #2.

For \( \frac{1}{4} < \omega < \frac{2}{10} \) the slope is \(+45 \text{ degrees/decade}\) so point #3.

\[
-90 + 45 \log \left( \frac{1}{20} \right) = -76.454
\]

For \( \frac{2}{10} < \omega < \frac{3}{10} \) the slope is \(+45 -45 = 0\) so point #4.

For \( \frac{3}{10} < \omega < 10 \cdot \frac{1}{4} \) the slope is \(+45 -45 +45 = 45\) so point #5.

\[
-76.454 + 45 \log \left( \frac{10}{4} \right) = -35.017
\]

For \( 10 \cdot \frac{1}{4} < \omega < 10 \cdot \frac{1}{2} \) the slope is \(-45 +45 = 0\) so point #6.

For \( 10 \cdot \frac{1}{2} < \omega < 30 \) the slope is \(+45\) so point #7.

\[
-35.017 + 45 \log \left( \frac{30}{10 \cdot \frac{1}{2}} \right) = 0.
\]

For \( \omega > 30 \) the slope is 0 so point #8.
Example #4

\[ H(s) = \frac{100000 \cdot s}{(s + 10000) \cdot (s + 50000)} = \frac{100000}{10000 \cdot 50000} \cdot \left( \frac{s}{10000} + 1 \right) \cdot \left( \frac{s}{50000} + 1 \right) \]

\[ \omega := 100, 200 \ldots 1000000 \]

\[ \Phi_1(\omega) := \begin{cases} 0 & \text{if } \omega < \frac{10000}{10} \\ 0 & \text{if } \omega < 10 \cdot 10000, -45 - 45 \log \left( \frac{\omega}{10000} \right), -90 \end{cases} \]

\[ \Phi_2(\omega) := \begin{cases} 0 & \text{if } \omega < \frac{50000}{10} \\ 0 & \text{if } \omega < 10 \cdot 50000, -45 - 45 \log \left( \frac{\omega}{50000} \right), -90 \end{cases} \]
\[
\begin{align*}
A := & \begin{pmatrix} 100 & 90 \\ 10000 & 90 \\ 10000 & 0 \\ 50000 & 0 \\ 50000 & -90 \\ 10^6 & -90 \end{pmatrix} \\
B := & \begin{pmatrix} 10000 & 90 \\ \frac{10000}{10} & 90 \\ 10 \cdot 10000 & 0 \end{pmatrix} \\
C := & \begin{pmatrix} 50000 & 90 \\ \frac{50000}{10} & 90 \\ 10 \cdot 50000 & 0 \end{pmatrix} \\
D := & \begin{pmatrix} 10000 & 90 \\ \frac{50000}{10} & 90 \\ 10 \cdot 10000 & 0 \\ 10 \cdot 50000 & 0 \end{pmatrix} \\
\Phi_T(\omega) := & 90 + \Phi_1(\omega) + \Phi_2(\omega) \\
\text{points :=} & \begin{pmatrix} 100 & 90 \\ \frac{10000}{10} & 90 \\ \frac{50000}{10} & 58.546 \\ 10000 \cdot 10 & -58.546 \\ 50000 \cdot 10 & -90 \\ 10^6 & -90 \end{pmatrix}
\end{align*}
\]
For $\omega < 1000$, the only contribution is from the zero at 0 so points #1 and #2.

For $1000 < \omega < 5000$, the slope is -45 deg per decade so point #3.

$90 - 45 \log \left( \frac{50000}{10} \right) = 58.546$

For $5000 < \omega < 10^5$, the slope is

$58.546 - 90 \cdot \log \left( \frac{10 \cdot 10000}{50000} \right) = -58.547$

For $10^5 < \omega < 5 \cdot 10^5$, the slope is -45 deg per decade so point #5.

$-58.547 - 45 \cdot \log \left( \frac{5 \cdot 10^5}{10^5} \right) = -90$

For $\omega > 5 \cdot 10^5$ the slope is zero, so point #6.
Second Order Magnitude Approximations

When we gave the general form of the transfer function as:

\[
H(s) = \frac{K \cdot \prod_{i=1}^{n_z} (s + \omega z_i)}{\prod_{i=1}^{n_p} (s + \omega p_i)} \quad \text{zero}
\]

\[
= \frac{K \cdot [(s + \omega z_1) \cdot (s + \omega z_2) \cdots (s + \omega z_{n_z})]}{(s + \omega p_1) \cdot (s + \omega p_2) \cdots (s + \omega p_{n_p})} \quad \text{pole}
\]

We allowed for the possibility that some of the constants \( \omega z_i \) and \( \omega p_i \) might be complex numbers as a result of second order terms that do not factor into real first order terms. To make magnitude and phase approximations of such terms, the terms are not factored into complex form but are left as second order terms of the form:

\[
\frac{s^2}{\omega_o^2} + 2 \cdot \frac{\zeta}{\omega_o} \cdot s + 1
\]

where \( \zeta < 1 \), since if \( \zeta = 1 \), the quadratic is a perfect square and is handled as a pole of multiplicity 2, and if \( \zeta > 1 \) the quadratic is factorable into two real factors and the term is handled as two distinct first order poles. The second order response has been normalized so that the constant coefficient is one. Note that in the following discussions \( \omega \) is a variable and \( \omega_o \) is the name of a constant.

**Magnitude Response of Second Order Poles**

Consider a transfer function of the form:

\[
H(s) = \frac{1}{\frac{s^2}{\omega_o^2} + 2 \cdot \frac{\zeta}{\omega_o} \cdot s + 1}
\]

\[
H(j \cdot \omega) = \frac{1}{1 - \frac{\omega^2}{\omega_o^2} + \frac{2 \cdot \zeta \cdot \omega}{\omega_o}}
\]

The low frequency asymptote is found by taking the limit as \( \omega \) approaches zero of \( H(j \omega) \).
\[ |H(j\omega)|_{dB} = \lim_{\omega \to 0} 20 \cdot \log \left( \frac{1}{\frac{-\omega^2}{\omega_0^2} + 2 \cdot \frac{\zeta}{\omega_0} \cdot j \cdot \omega + 1} \right) = 0 \text{ dB} \]

The high frequency asymptote is found by taking the limit as \( \omega \) approaches infinity of \( H(j\omega) \).

\[ |H(j\omega)|_{dB} = \lim_{\omega \to \infty} 20 \cdot \log \left( \frac{1}{\frac{-\omega^2}{\omega_0^2} + 2 \cdot \frac{\zeta}{\omega_0} \cdot j \cdot \omega + 1} \right) \]

\[ = \lim_{\omega \to \infty} 20 \cdot \log \left( \frac{1}{\frac{-\omega^2}{\omega_0^2}} \right) = \lim_{\omega \to \infty} 20 \cdot \log \left( \frac{\omega_0^2}{\omega^2} \right) \]

\[ = \lim_{\omega \to \infty} \left( 20 \cdot \log \left( \omega_0^2 \right) - 20 \cdot \log \left( \omega^2 \right) \right) \]

\[ = -20 \cdot \log \left( \omega^2 \right) = -40 \cdot \log(\omega) \]

The approximation strategy is the same as with first order magnitude plots: The magnitude function is approximated by a horizontal straight line of 0 dB for frequencies less than \( \omega_0 \) and by a straight line of slope -40 dB per decade passing through \( \omega = \omega_0 \) for all frequencies greater than \( \omega_0 \). The main difference between the first order case and the second order case, is that the error depends heavily on \( \zeta \) and may not usually be ignored. Several second order magnitude functions are plotted below for various values of \( \zeta \) (\( \omega_0 \) has been normalized to 1.0).

\[ H_a(\omega) := 20 \cdot \log \left[ \frac{1}{\left| (1 - \omega^2) + [2 \cdot \omega \cdot (1) \cdot i] \right|} \right] \quad (\zeta = 1.0) \]
\[ H_b(\omega) := 20 \cdot \log \left[ \frac{1}{(1 - \omega^2) + \left[ 2 \cdot \omega \cdot \left( \frac{1}{\sqrt{2}} \right) \right] i} \right] \] \quad \left( \zeta = \frac{1}{\sqrt{2}} \right)

\[ H_c(\omega) := 20 \cdot \log \left[ \frac{1}{(1 - \omega^2) + \left[ 2 \cdot \omega \cdot \left( \frac{6}{10} \right) \right] i} \right] \] \quad \left( \zeta = \frac{6}{10} \right)

\[ H_e(\omega) := 20 \cdot \log \left[ \frac{1}{(1 - \omega^2) + \left[ 2 \cdot \omega \cdot \left( \frac{1}{2} \right) \right] i} \right] \] \quad \left( \zeta = \frac{1}{2} \right)

\[ H_d(\omega) := 20 \cdot \log \left[ \frac{1}{(1 - \omega^2) + \left[ 2 \cdot \omega \cdot \left( \frac{1}{5} \right) \right] i} \right] \] \quad \left( \zeta = \frac{1}{5} \right)

\[ H_f(\omega) := 20 \cdot \log \left[ \frac{1}{(1 - \omega^2) + \left[ 2 \cdot \omega \cdot \left( \frac{1}{10} \right) \right] i} \right] \] \quad \left( \zeta = \frac{1}{10} \right)

\[ H_g(\omega) := 20 \cdot \log \left[ \frac{1}{(1 - \omega^2) + \left[ 2 \cdot \omega \cdot \left( \frac{1}{100} \right) \right] i} \right] \] \quad \left( \zeta = \frac{1}{100} \right)

\[ \omega := 0.1, 0.15..100 \]
The above plot has been expanded around $\omega = \omega_0$ to expose the three distinct characteristics that such transfer functions exhibit, depending on the value of $\zeta$, and is shown on the next page.
We see that for \( 1 > \zeta > \frac{1}{\sqrt{2}} \) the response does not peak. (Ha and Hb)

The case \( \zeta = \frac{1}{\sqrt{2}} \) is the Butterworth maximally flat response. (Hb)

For all \( \zeta \) such that \( 1 > \zeta > \frac{1}{\sqrt{2}} \) the maximum of the magnitude response occurs at \( \omega = 0 \).

For values of \( \zeta \) such that \( \frac{1}{\sqrt{2}} > \zeta > \frac{1}{2} \) the magnitude function peaks, but returns to 0 dB at a frequency smaller than \( \omega_o \). (Hc) For values of \( \zeta \) such that \( \frac{1}{2} > \zeta \) the response peaks and returns to 0 dB at a frequency larger than \( \omega_o \). (He, Hf,Hg)

For \( \zeta = \frac{1}{2} \) the response peaks and returns to 0 dB at \( \omega = \omega_o \). (Hd)
Peaked Responses

For values of $\zeta$ such that $\zeta < \frac{1}{\sqrt{2}}$, we define $\omega_o$, $\omega_p$, $\omega_c$, $\omega_2$, $A_o$, $A_p$, and $A_2$ as shown in the plot below.

$$H_p(\omega) := 20 \cdot \log \left[ \left| \frac{1}{(1 - \omega^2) + \left( 2 \cdot \omega \cdot \left( \frac{3}{10} \right) i \right)} \right| \right]$$

($\omega_o = 1$ in this example)

$A_o$ is the magnitude of the response at $\omega = \omega_o$, $A_p$ is the magnitude at $\omega_p$ where the response peaks, the response crosses through 0 dB at $\omega_c$, $\omega_2$ is one half $\omega_o$, and $A_2$ is the magnitude of the response at $\omega_2$.

$A_p := 4.85 \quad A_o := 4.4 \quad A_2 := 1.9$ (We’ll give formulae for these presently.)
Derivation of $\omega_p$, $\omega_c$, $\omega_2$, $A_o$, $A_p$, and $A_2$

\[
H(s) = \frac{1}{\left(\frac{s^2}{\omega_0^2} + \frac{2\cdot\zeta\cdot s}{\omega_0} + 1\right)}
\]

\[
(\|H(j\omega)\|^2) = \frac{1}{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \left(\frac{2\cdot\zeta\cdot\omega}{\omega_0}\right)^2}
\]

Now $|H(j\omega)|$ reaches a peak at $\omega = \omega_p$. Therefore:

\[
\frac{d}{d\omega}\left[\frac{1}{\left(\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \left(\frac{2\cdot\zeta\cdot\omega}{\omega_0}\right)^2\right)}\right]_{\omega = \omega_p} = 0
\]

\[
= \left[2\cdot\left(1 - \frac{\omega^2}{\omega_0^2}\right)\cdot\left(-\frac{2\cdot\omega}{\omega_0}\right) + 2\cdot\left(\frac{2\cdot\zeta\cdot\omega}{\omega_0}\right)\cdot\left(\frac{2\cdot\zeta}{\omega_0}\right)\right]_{\omega = \omega_p}
\]

The value of $\omega$ that satisfies this equation is $\omega_p$

In order for this function to equal zero the numerator must equal zero (there is no combination of positive $\omega$, $\omega_o$, and $\zeta$ that could produce a zero denominator), so:

\[
2\cdot\left(1 - \frac{\omega_p^2}{\omega_0^2}\right)\cdot\left(-\frac{2\cdot\omega_p}{\omega_0}\right) + 2\cdot\left(\frac{2\cdot\zeta\cdot\omega_p}{\omega_0}\right)\cdot\left(\frac{2\cdot\zeta}{\omega_0}\right) = -\frac{4\cdot\omega_p^2}{\omega_0^2} + \frac{4\cdot\omega_p^3}{\omega_0^4} + \frac{8\cdot\zeta^2\cdot\omega_p}{\omega_0^2}
\]

\[
= -1 + \frac{\omega_p^2}{\omega_0^2} + 2\cdot\zeta^2 = 0
\]
Therefore, the response peaks at a frequency $\omega_p$ such that:

$$\omega_p = \omega_o \sqrt{1 - 2 \cdot \zeta^2}$$

The magnitude crosses back through 0 dB at $\omega = \omega_c$, so

$$20 \cdot \log \left( |H(j\omega)| \right) = 0,$$

and

$$|H(j\omega)| = 1 \quad \text{so} \quad \left( |H(j\omega)| \right)^2 = 1.$$ As a result we have:

$$1 = \frac{1}{\left( 1 - \frac{\omega_c^2}{\omega_o^2} \right)^2 + \left( \frac{2 \cdot \zeta \cdot \omega_c}{\omega_o} \right)^2}$$

$$1 - \left( 2 \cdot \frac{\omega_c^2}{\omega_o^2} \right) + \frac{\omega_c^4}{\omega_o^4} + \frac{4 \cdot \zeta^2 \cdot \omega_c^2}{\omega_o^2} = 1 \quad \text{So}$$

$$\left( -2 \cdot \frac{\omega_c^2}{\omega_o^2} \right) + \frac{\omega_c^4}{\omega_o^4} + \frac{4 \cdot \zeta^2 \cdot \omega_c^2}{\omega_o^2} = 0 \quad \text{and}$$

$$\frac{\omega_c^2}{\omega_o^2} - 2 + 4 \cdot \zeta^2 = 0 \quad \text{Or} \quad \omega_c = \omega_o \sqrt{2 \cdot (1 - 2 \cdot \zeta^2)} = \sqrt{2} \cdot \omega_p$$

Since $\omega_p = \omega_o \sqrt{1 - 2 \cdot \zeta^2}$

The transfer function has a magnitude $A_o$ at $\omega = \omega_o$, so

$$A_o = 20 \cdot \log \left| \frac{1}{1 - \frac{\omega_o^2}{\omega_o^2} + i \left( \frac{2 \cdot \zeta \cdot \omega_o}{\omega_o} \right)} \right|$$
\[
20 \cdot \log \left[ \sqrt{\frac{1}{\left(1 - \frac{\omega_p^2}{\omega_o^2}\right)^2 + \left(\frac{2\cdot\zeta \cdot \omega_p}{\omega_o}\right)^2}}\right]
\]

\[
= 20 \cdot \log(1) - 20 \cdot \log \left[ \sqrt{\frac{\omega_o^2}{\left(1 - \frac{\omega_p^2}{\omega_o^2}\right)^2 + \left(\frac{2\cdot\zeta \cdot \omega_p}{\omega_o}\right)^2}}\right]
\]

\[
= -20 \cdot \log \left[ \sqrt{0 + 4 \cdot \zeta^2} \right] = -20 \cdot \log(2 \cdot \zeta) = A_o
\]

At \(\omega = \omega_p\) the magnitude is:

\[
A_p = 20 \cdot \log \left[ \sqrt{\frac{1}{\left(1 - \frac{\omega_p^2}{\omega_o^2}\right)^2 + i \cdot \left(\frac{2\cdot\zeta \cdot \omega_p}{\omega_o}\right)^2}}\right]
\]

\[
= 20 \cdot \log \left[ \sqrt{\frac{1}{\left(1 - \frac{\omega_p^2}{\omega_o^2}\right)^2 + \frac{2\cdot\zeta \cdot \omega_p}{\omega_o} \cdot \sqrt{1 - 2 \cdot \zeta^2}} \right]
\]

\[
= 20 \cdot \log(1) - 20 \cdot \log \left[ \sqrt{\frac{\omega_o^2}{\left(1 - \frac{\omega_p^2}{\omega_o^2}\right)^2 + \frac{2\cdot\zeta \cdot \omega_p}{\omega_o} \cdot \sqrt{1 - 2 \cdot \zeta^2}}} \right]
\]

\[
= 20 \cdot \log \left[ \sqrt{\frac{\omega_o^2}{\left(1 - \frac{\omega_p^2}{\omega_o^2}\right)^2 + \frac{2\cdot\zeta \cdot \omega_p}{\omega_o} \cdot \sqrt{1 - 2 \cdot \zeta^2}}} \right]
\]
\[ A_p = -10 \log \left[ 1 - \left( \frac{\omega_o \cdot \sqrt{1 - 2 \cdot \zeta^2}}{\omega_o^2} \right)^2 \right] + \left[ \frac{2 \cdot \zeta \cdot \left( \frac{\omega_o \cdot \sqrt{1 - 2 \cdot \zeta^2}}{\omega_o} \right)^2}{\omega_o} \right] \]

\[ = -10 \log \left[ 1 - \left( 1 - 2 \cdot \zeta^2 \right) \right]^2 + 4 \cdot \zeta^2 \cdot (1 - 2 \cdot \zeta^2) \]

\[ = -10 \log \left( 4 \cdot \zeta^4 + 4 \cdot \zeta^2 - 8 \cdot \zeta^4 \right) = -10 \log \left[ 4 \cdot \zeta^2 \cdot (1 - \zeta^2) \right] \]

So \[ A_p = -10 \log \left[ 4 \cdot \zeta^2 \cdot (1 - \zeta^2) \right] \]

When \( \omega = \frac{\omega_o}{2} \) the magnitude is:

\[ A_2 = 20 \cdot \log \left[ 1 - \left( \frac{\omega_o}{2} \right)^2 + i \cdot \left( \frac{2 \cdot \zeta \cdot \omega_o}{\omega_o} \right) \right] \]

\[ = -20 \cdot \log \left[ 1 - \left( \frac{\omega_o}{2} \right)^2 \right]^2 + \left( \frac{2 \cdot \zeta \cdot \omega_o}{\omega_o} \right)^2 \]

\[ = -10 \log \left( \frac{9}{16} + \zeta^2 \right) = -10 \log \left( \zeta^2 + 0.5625 \right) \]
In summary for $\zeta < \frac{1}{\sqrt{2}}$ we have: $\omega_p = \omega_o \sqrt{1 - 2 \cdot \zeta^2}$ $\omega_c = \sqrt{2} \cdot \omega_p$

$A_o = -20 \cdot \log(2 \cdot \zeta)$ $A_p = -10 \cdot \log \left[ 4 \cdot \zeta^2 \cdot (1 - \zeta^2) \right]$ $A_2 = -10 \cdot \log \left( \zeta^2 + 0.5625 \right)$

For example when $\zeta = 0.3$ and $\omega_o = 1$ (This is the value of $\zeta$ used in the illustration of the definition of $\omega_o, \omega_p, \omega_c, A_o, A_p, A_2$), we obtain $\omega_p = 0.9055$, $\omega_c = 1.281$, $A_o = 4.437$ dB, $A_p = 4.847$ dB, and $A_2 = 1.854$ dB.

Non-peaked Responses

For non-peaked responses, i.e. when $\zeta > \frac{1}{\sqrt{2}}$

$\omega_p$, $A_p$, and $\omega_c$ are not defined so we plot only $A_o$ and $A_c$. For example if $\omega_o = 1$ and $\zeta = 0.8$, we have:

$$H_{np}(\omega) := 20 \cdot \log \left[ \left( 1 - \omega^2 \right) + \left( 2 \cdot \omega \cdot \left( \frac{8}{10} \right) \cdot i \right) \right]$$

$$A_0 := -20 \cdot \log(2 \cdot 0.8)$$

$$A_2 := -10 \cdot \log \left( 0.8^2 + 0.563 \right)$$
**Extraction of $\zeta$ and $\omega_0$ from polynomial coefficients**

Suppose we are given the polynomial $s^2 + a \cdot s + b$ and we wish to calculate $\zeta$ and $\omega_0$.

Since $s^2 + a \cdot s + b = s^2 + 2 \cdot \zeta \cdot \omega_0 + \omega_0^2$ we have

$$\omega_0 = \sqrt{b} \quad \text{and} \quad 2 \cdot \zeta \cdot \omega_0 = a = 2 \cdot \zeta \cdot \sqrt{b} \quad \text{so} \quad \zeta = \frac{a}{2 \cdot \sqrt{b}}$$

Example #1: suppose we are given the transfer function $H(s) = \frac{1}{s^2 + 2 \cdot s + 25}$

Find the Bode magnitude plot.

$$a := 2 \quad b := 25 \quad \omega_0 := \sqrt{b} \quad \omega_0 = 5 \quad \zeta := \frac{a}{2 \cdot \sqrt{b}} \quad \zeta = 0.2$$

$$H(s) = \frac{1}{25} \cdot \frac{1}{\left(s^2 + \frac{2}{25} \cdot s + 1\right)}$$

(Notice that the normalization does not affect the values of $\omega_0$ and $\zeta$, so $\omega_0$ and $\zeta$ can be calculated before the normalization.)

After the normalization we have $H(s) = \frac{s^2}{b} + \frac{a}{b} \cdot s + 1 = \frac{s^2}{\omega_0^2} + \frac{2 \cdot \zeta}{\omega_0} \cdot s + 1$ and we get precisely the same formulae for $\omega_0$ and $\zeta$ as before, so the calculation of $\omega_0$ and $\zeta$ can be carried out either before or after the normalization.

$$\frac{\omega_0}{2} = 2.5 \quad \omega_p := \omega_0 \cdot \sqrt{1 - 2 \cdot \zeta^2} \quad \omega_p = 4.796 \quad \omega_c := \sqrt{2} \cdot \omega_p \quad \omega_c = 6.782$$

$$A_0 := -20 \cdot \log(2 \cdot \zeta) \quad A_0 = 7.959 \quad A_p := -10 \cdot \log\left[4 \cdot \zeta^2 \cdot (1 - \zeta^2)\right] \quad A_p = 8.136$$

$$A_2 := -10 \cdot \log\left(\zeta^2 + 0.5625\right) \quad A_2 = 2.2$$

The resulting plot is shown below.

$$\omega := 0.1, 0.15 \ldots 100$$
\[
\begin{align*}
\text{Corr} := & \begin{pmatrix}
\frac{\omega_0}{10} & K_0 \\
\frac{\omega_0}{2} & K_0 + A_2 \\
\omega_p & K_0 + A_p \\
\omega_0 & K_0 + A_0 \\
\omega_c & K_0 \\
10 \cdot \omega_0 & K_0 - 40
\end{pmatrix} \\
K_0 := & 20 \cdot \log \left( \frac{1}{25} \right) \\
K_0 = & -27.959 \\
L(\omega) := & \begin{cases}
\omega < \omega_0, & K_0 - 40 \cdot \log \left( \frac{\omega}{\omega_0} \right) \\
\omega_0, & K_0 - 40 \cdot \log \left( \frac{\omega}{\omega_0} \right)
\end{cases} \\
H(\omega) := & 20 \cdot \log \left[ \frac{1}{25} \cdot \left( 1 - \frac{\omega^2}{\omega_0^2} \right) + j \cdot 2 \cdot \zeta \cdot \frac{\omega}{\omega_0} \right]
\end{align*}
\]

Note that \( A_p \) and \( A_0 \) are too close to resolve. This is usually the case for small values of \( \zeta \).
Example #2 \( H(s) = \frac{1}{s^2 + 0.36 + 4} \)

\[ \omega_0 := \sqrt{4} \quad \omega_0 = 2 \quad \zeta := \frac{0.36}{2\sqrt{4}} \quad \zeta = 0.09 \]

\[ \frac{\omega_0}{2} = 1 \quad \omega_p := \omega_0 \sqrt{1 - 2 \cdot \zeta^2} \quad \omega_p = 1.984 \quad \omega_c := \sqrt{2} \cdot \omega_p \quad \omega_c = 2.805 \]

\[ A_0 := -20 \cdot \log(2 \cdot \zeta) \quad A_0 = 14.895 \quad A_p := -10 \cdot \log\left[4 \cdot \zeta^2 \cdot (1 - \zeta^2)\right] \quad A_p = 14.93 \]

\[ K_0 := 20 \cdot \log\left(\frac{1}{4}\right) \quad K_0 = -12.041 \]

\[ A_2 := -10 \cdot \log\left(\zeta^2 + 0.5625\right) \quad A_2 = 2.437 \]

\[ L(\omega) := \text{if} \left(\omega < \omega_0, K_0, K_0 - 40 \cdot \log\left(\frac{\omega}{\omega_0}\right)\right) \]

\[ H(\omega) := 20 \cdot \log \left[ \frac{1}{4} \cdot \left( \frac{1}{1 - \frac{\omega^2}{\omega_0^2}} + j \cdot \frac{2 \cdot \zeta \cdot \omega}{\omega_0} \right) \right] \]
Example #3  \( H(s) = \frac{1}{s^2 + 8.4s + 49} \)

\[
\omega_0 := \sqrt{49} \quad \omega_0 = 7 \quad \zeta := \frac{8.4}{2 \cdot \sqrt{49}} \quad \zeta = 0.6
\]

\[
\frac{\omega_0}{2} = 3.5 \quad \omega_p := \omega_0 \sqrt{1 - 2 \cdot \zeta^2} \quad \omega_p = 3.704 \quad \omega_c := \sqrt{2 \cdot \omega_p} \quad \omega_c = 5.238
\]

\[
A_0 := -20 \cdot \log(2 \cdot \zeta) \quad A_0 = -1.584 \quad A_p := -10 \cdot \log\left[4 \cdot \zeta^2 \cdot (1 - \zeta^2)\right] \quad A_p = 0.355
\]

Note that since \( \omega_c < \omega_0 \), their order in Corr is reversed.

\[
A_2 := -10 \cdot \log\left(\zeta^2 + 0.5625\right) \quad A_2 = 0.35
\]

\[
K_0 := 20 \cdot \log\left(\frac{1}{49}\right) \quad K_0 = -33.804
\]

\[
L(\omega) := \text{if}\left(\omega < \omega_0, K_0, K_0 - 40 \cdot \log\left(\frac{\omega}{\omega_0}\right)\right)
\]

\[
H(\omega) := 20 \cdot \log\left[\frac{1}{49} \cdot \left|1 - \frac{\omega^2}{\omega_0^2}\right| + j \cdot \left(2 \cdot \zeta \cdot \frac{\omega}{\omega_0}\right)\right]
\]
Example #4 \( H(s) = \frac{1}{s^2 + 12.8 \cdot s + 64} \quad \omega_0 := \sqrt{64} \quad \omega_0 = 8 \quad \zeta := \frac{12.8}{2 \cdot \sqrt{64}} \quad \zeta = 0.8 \)

\[ \frac{\omega_0}{2} = 4 \quad \text{Since} \quad \zeta > \frac{1}{\sqrt{2}}, \quad \omega_p, \omega_c, A_p \text{ are not defined} \]

\[ A_0 := -20 \cdot \log(2 \cdot \zeta) \quad A_0 = -4.082 \quad A_2 := -10 \cdot \log(\zeta^2 + 0.5625) \quad A_2 = -0.801 \]

\[
\begin{align*}
\text{Corr} := \begin{pmatrix}
\omega_0 \\
\omega_0 \\
10 \cdot \omega_0
\end{pmatrix}
\begin{pmatrix}
\frac{\omega_0}{10} & K_0 \\
\omega_0 & K_0 + A_2 \\
10 \cdot \omega_0 & K_0 - 40
\end{pmatrix}
\end{align*}
\]

\[ K_0 := 20 \cdot \log \left( \frac{1}{64} \right) \quad K_0 = -36.124 \]

\[ L(\omega) := \begin{cases} 
\omega < \omega_0, K_0, K_0 - 40 \cdot \log \left( \frac{\omega}{\omega_0} \right) 
\end{cases} \]

\[ H(\omega) := 20 \cdot \log \left[ \frac{1}{64} \cdot \left( \frac{1}{1 - \frac{\omega^2}{\omega_0^2}} \right) + j \cdot \left( 2 \cdot \frac{\zeta \cdot \omega}{\omega_0} \right) \right] \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{diagram.png}
\caption{Graph of \( L(\omega) \), Corr, and \( H(\omega) \) versus \( \omega \) with lines at \( \omega_0 = \sqrt{64} \) and \( \omega_c = 2 \cdot \sqrt{64} \).}
\end{figure}
Example #5 \( H(s) = \frac{1}{s^2 + 4s + 3} \quad \zeta := \frac{4}{2 \cdot \sqrt{3}} \quad \zeta = 1.155 \)

Since \( \zeta > 1 \), \( H(s) \) can be factored into two real factors.

\[
s_1 := \frac{-4}{2} + \frac{\sqrt{16 - 12}}{2} \quad s_1 = -1
\]

\[
s_2 := \frac{-4}{2} - \frac{\sqrt{16 - 12}}{2} \quad s_2 = -3
\]

So \( \frac{1}{s^2 + 4s + 3} = \frac{1}{(s + 1)(s + 3)} \)

\[M_1(\omega) := \text{if } \omega < 1, 0, -20 \log \left( \frac{\omega}{1} \right)\]

\[M_2(\omega) := \text{if } \omega < 3, 0, -20 \log \left( \frac{\omega}{3} \right)\]

\[M_T(\omega) := M_1(\omega) + M_2(\omega)\]

![Graph of \( M_T(\omega) \) vs. \( \omega \)]
Magnitude Response of Second Order Zeros

Transfer functions of the form \( H(s) = \frac{s^2}{\omega_0^2} + \frac{2 \cdot \zeta}{\omega_0} \cdot s + 1 \) (i.e. second order zeros)

are handled in a manner similar to the approach we used on second order poles. The values of \( \omega_0, \omega_c, \) and \( \omega_p \) are as before, and

\[
A_0 := 20 \cdot \log(2 \cdot \zeta) \quad \omega_0 := \sqrt{25} \quad \omega_0 = 5
\]

\[
A_p := 10 \cdot \log\left[ 4 \cdot \zeta^2 \cdot (1 - \zeta^2) \right] \quad A_2 := 10 \cdot \log(\zeta + 0.5625)
\]

Note the change in sign for all the magnitudes.

Example #1 \( H(s) = s^2 + 2 \cdot s + 25 \quad \omega_0 := \sqrt{25} \quad \zeta := \frac{2}{2 \cdot \sqrt{25}} \)

\[
\omega_0 = \frac{2.5}{2} \quad \omega_p := \omega_0 \cdot \sqrt{1 - 2 \cdot \zeta^2} \quad \omega_p = 4.796 \quad \omega_c := \sqrt{2} \cdot \omega_p \quad \omega_c = 6.782
\]

\[
A_0 := 20 \cdot \log(2 \cdot \zeta) \quad A_0 = -7.959 \quad A_p := 10 \cdot \log\left[ 4 \cdot \zeta^2 \cdot (1 - \zeta^2) \right] \quad A_p = -8.136
\]

\[
A_2 := 10 \cdot \log\left( \zeta^2 + 0.5625 \right) \quad A_2 = -2.2 \quad H(s) = 25 \left( \frac{s^2}{25} + \frac{2}{25} \cdot s + 1 \right)
\]

\[
\omega := 0.1, 0.15 \ldots 100 \quad K_0 := 20 \cdot \log(25) \quad K_0 = 27.959
\]
Example #2  \( H(s) = s^2 + 12.8 \cdot s + 64 \) \( \omega_0 := \sqrt{64} \) \( \omega_0 = 8 \) \( \zeta := \frac{12.8}{2 \cdot \sqrt{64}} \) \( \zeta = 0.8 \)

\[ \frac{\omega_0}{2} = 4 \quad \text{Since } \zeta > \frac{1}{\sqrt{2}}, \ \omega_p, \omega_c, A_p \text{ are not defined} \]

\[ A_0 := 20 \cdot \log(2 \cdot \zeta) \quad A_0 = 4.082 \quad A_2 := 10 \cdot \log(\zeta^2 + 0.5625) \]

\[ \text{Corr} := \begin{pmatrix} \frac{\omega_0}{10} & K_0 \\ \frac{\omega_0}{2} & K_0 + A_2 \\ \omega_0 & K_0 + A_0 \\ 10 \cdot \omega_0 & K_0 + 40 \end{pmatrix} \]

\[ K_0 := 20 \cdot \log(64) \quad K_0 = 36.124 \]

\[ L(\omega) := \text{if} \left( \omega < \omega_0, K_0, K_0 + 40 \cdot \log\left(\frac{\omega}{\omega_0}\right) \right) \]

\[ H(\omega) := 20 \cdot \log \left[ 64 \cdot \left( 1 - \frac{\omega^2}{\omega_0^2} \right) + j \cdot 2 \cdot \zeta \cdot \frac{\omega}{\omega_0} \right] \]
Second Order Phase Approximations

The second order phase response is the difference in phase between the input and the output sinusoids of a system characterized by a transfer function of the form:

\[ H(s) = \frac{s^2 + 2 \cdot \zeta \cdot \omega_0 \cdot s + \omega_0^2}{s^2 + 2 \cdot \zeta \cdot \omega_0 \cdot s + \omega_0^2} \]

The phase function is \( \Phi(\omega) = \text{atan}\left(\frac{2 \cdot \zeta \cdot \omega_0 \cdot \omega}{\omega_0^2 - \omega^2}\right) \)

First Straight-Line Approximation (Inadequate)

We begin as with the first order case by finding the slope of the phase function at \( \omega = \omega_0 \) and passing a straight line with that slope through the phase function at \( \omega = \omega_0 \). The slope of the phase function is:

\[
\frac{d}{d\omega} \Phi(\omega) = \left[ \frac{2 \cdot \zeta \cdot \frac{\omega_0}{\omega_0^2 - \omega^2} + 4 \cdot \zeta \cdot \omega_0 \cdot \frac{\omega^2}{(\omega_0^2 - \omega^2)^2}}{1 + 4 \cdot \zeta^2 \cdot \omega_0^2 \cdot \frac{\omega^2}{(\omega_0^2 - \omega^2)^2}} \right]
\]
which simplifies to
\[
2 \cdot \zeta \cdot \omega_0 \cdot \left( \frac{\omega_0^2 + \omega^2}{\omega_0^4 - 2 \cdot \omega_0^2 \cdot \omega^2 + \omega^4 + 4 \cdot \zeta^2 \cdot \omega_0^2 \cdot \omega^2} \right)
\]

Evaluating this expression at \( \omega = \omega_o \) gives the slope of \( \Phi(\omega) \) at \( \omega = \omega_o \) as
\[
\frac{1}{(\zeta \cdot \omega_0)} \text{ radians per radian}
\]

To convert a slope of \( k \) radians per radian to an equivalent slope in radians per decade we proceed as follows: the rise is the limit as \( \Delta \to 0 \) of
\[
k \cdot \left[ \left( \omega + \Delta \right) - \left( \omega - \Delta \right) \right]
\]
and the run is the limit as \( \Delta \to 0 \) of \( \log \left( \frac{\omega_o + \Delta}{\omega_o - \Delta} \right) \). The slope in radians per decade is then:
\[
\lim_{\Delta \to 0} \frac{k \cdot \left[ \left( \omega_o + \Delta \omega \right) - \left( \omega_o - \Delta \omega \right) \right]}{\log \left( \frac{\omega_o + \Delta \omega}{\omega_o - \Delta \omega} \right)} = k \cdot \omega_0 \cdot (\ln(2) + \ln(5)) = \frac{k \cdot \omega_o}{\log(e)}
\]

In our case the slope in radians per radian is \( k = \frac{1}{(\zeta \cdot \omega_0)} \) which gives the slope at \( \omega = \omega_o \) as
\[
\frac{1}{\zeta \cdot \log(e)} \text{ radians per decade or as } \frac{360}{2 \cdot \pi \cdot \zeta \cdot \log(e)} = \frac{131.92841}{\zeta} \text{ degrees per decade}
\]

The slope of the phase function \( \Phi(\omega) \) at \( \omega = \omega_0 \) is independent of \( \omega_o \) but depends on \( \zeta \).

Phase functions for various values of \( \zeta \) are shown below along with the approximation obtained by extending a line of slope \( 132 / \zeta \) degrees per decade through the point on the phase function where \( \omega = \omega_0 \). (We have normalized \( \omega_o \) to 1.)
\[\omega_0 := 1 \quad \zeta := 1\]

\[
\arg\left[2 \cdot \zeta \cdot \omega_0 \cdot \omega \cdot \left(i \left(\omega_0^2 - \omega^2\right)\right)\right] \cdot \frac{360}{2 \cdot \pi}
\]

\[90 + \frac{132}{\zeta} \cdot \log(\omega)\]

\[\zeta := 0.8\]

\[
\arg\left[2 \cdot \zeta \cdot \omega_0 \cdot \omega \cdot \left(i \left(\omega_0^2 - \omega^2\right)\right)\right] \cdot \frac{360}{2 \cdot \pi}
\]

\[90 + \frac{132}{\zeta} \cdot \log(\omega)\]

\[\zeta := 0.6\]

\[
\arg\left[2 \cdot \zeta \cdot \omega_0 \cdot \omega \cdot \left(i \left(\omega_0^2 - \omega^2\right)\right)\right] \cdot \frac{360}{2 \cdot \pi}
\]

\[90 + \frac{132}{\zeta} \cdot \log(\omega)\]
\[ \zeta := 0.4 \]

\[ \arg \left[ 2 \cdot \zeta \cdot \omega_0 \cdot \omega \cdot i \left( \omega_0^2 - \omega^2 \right) \right] \cdot \frac{360}{2 \cdot \pi} \]

\[ 90 + \frac{132}{\zeta} \cdot \log(\omega) \]

\[ \zeta := 0.2 \]

\[ \arg \left[ 2 \cdot \zeta \cdot \omega_0 \cdot \omega \cdot i \left( \omega_0^2 - \omega^2 \right) \right] \cdot \frac{360}{2 \cdot \pi} \]

\[ 90 + \frac{132}{\zeta} \cdot \log(\omega) \]

\[ \zeta := 0.1 \]

\[ \arg \left[ 2 \cdot \zeta \cdot \omega_0 \cdot \omega \cdot i \left( \omega_0^2 - \omega^2 \right) \right] \cdot \frac{360}{2 \cdot \pi} \]

\[ 90 + \frac{132}{\zeta} \cdot \log(\omega) \]
As can be seen from the above plots (and as we shall show by direct calculation), this straight-line approximation suffers from the same shortcomings that made it useless for the first order case. To demonstrate the large errors at $\omega_{lo}$ and $\omega_{hi}$, we begin by calculating $\omega_{lo}$ the value of $\omega$ at which the straight line crosses 0°, and $\omega_{hi}$ the value of $\omega$ at which the straight line crosses 180°. $\omega_{lo}$ is the frequency where

$$90 + \frac{1}{\zeta \cdot \log(e)} \cdot \frac{360}{2 \cdot \pi} \cdot \log \left( \frac{\omega}{\omega_o} \right) = 0$$

Then

we have$$\frac{-90 \cdot 2 \cdot \pi \cdot \zeta \cdot \log(e)}{360} = \log \left( \frac{\omega_{lo}}{\omega_o} \right) = \left( \frac{-\pi}{2} \right) \cdot \zeta \cdot \log(e) = \log \left( e^{-\frac{\pi}{2} \cdot \zeta} \right)$$

so $\omega_{lo} = \omega_o \cdot \left( e^{-\frac{\pi}{2} \cdot \zeta} \right)$

$\omega_{lo} = \omega_o \cdot 4.81^{-\zeta}$

Similarly for $\omega_{hi}$ we have $\omega_{hi} = \omega_o \cdot \left( e^{\frac{\pi}{2} \cdot \zeta} \right) = \omega_o \cdot 4.81^{\zeta}$

The error at $\omega_{lo}$ is given by substitution of $\omega_{lo}$ into the expression for $\Phi(\omega)$, which gives

$$\text{Error}_{lo} = \text{atan} \left[ \frac{2 \cdot \zeta \cdot \omega_o \cdot \omega_o \cdot \left( \frac{\pi}{e^2} \right)^{-\zeta}}{\omega_o^2 - \left[ \omega_o \cdot \left( \frac{\pi}{e^2} \right)^{-\zeta} \right]^2} \right] = -\text{atan} \left[ 2 \cdot \zeta \cdot \frac{\exp \left( -\frac{1}{2} \cdot \pi \cdot \zeta \right)}{(-1 + \exp(-\pi \cdot \zeta))} \right]$$

The errors (in degrees) at $\omega_{lo}$ are shown below for various values of $\zeta$.

$\zeta := 0.1, 0.2, \ldots, 1.0$

$$\text{Error}_{lo}(\zeta) := -\text{atan} \left[ 2 \cdot \zeta \cdot \frac{\exp \left( -\frac{1}{2} \cdot \pi \cdot \zeta \right)}{(-1 + \exp(-\pi \cdot \zeta))} \right] \cdot \frac{360}{2 \cdot \pi}$$
\[ \text{Error}_{lo}(\zeta) = \]

\[
\begin{align*}
32.375 \\
32.058 \\
31.536 \\
30.82 \\
29.924 \\
28.868 \\
27.672 \\
26.359 \\
24.955 \\
23.487
\end{align*}
\]

Similarly for the error at \( \omega_{hi} \) we have: \( \text{Error}_{hi} = \)

\[
\frac{\pi}{2} - \frac{\pi}{2} + \tan^{-1} \left( \frac{2 \cdot \zeta \cdot \omega_0 \cdot \omega_0 \cdot \left( \frac{\pi}{e^2} \right)^\zeta}{\omega_0^2 \left( \omega_0 \cdot \left( \frac{\pi}{e^2} \right)^\zeta \right)^2} \right) = \tan^{-1} \left( \frac{\exp \left( \frac{1}{2} \cdot \pi \cdot \zeta \right)}{\left( -1 + \exp(\pi \cdot \zeta) \right)} \right)
\]

\[
\text{Error}_{hi}(\zeta) := \tan^{-1} \left( \frac{\exp \left( \frac{1}{2} \cdot \pi \cdot \zeta \right)}{\left( -1 + \exp(\pi \cdot \zeta) \right)} \right) \cdot \frac{360}{2 \cdot \pi}
\]

\[
\text{Error}_{hi}(\zeta) = \]

\[
\begin{align*}
32.375 \\
32.058 \\
31.536 \\
30.82 \\
29.924 \\
28.868 \\
27.672 \\
26.359 \\
24.955 \\
23.487
\end{align*}
\]
To extend our results for the first order phase approximation to the second order case, let us consider the second order transfer function with $\zeta = 1$. Proceeding as before we set $\omega_{hi}$ to $10 \omega_0$ and $\omega_{lo}$ to $\frac{10}{\omega_0}$, giving a straight line with slope $90^\circ$ per decade shown below along with the $\frac{132}{\zeta}$ degrees per decade line.

\[ \zeta := 1 \]

\[ \arg\left[2 \cdot \zeta \cdot \omega_0 \cdot \omega \cdot i + \left(\omega_0^2 - \omega^2\right)\right] \cdot \frac{360}{2 \cdot \pi} \]

\[ 90 + \frac{132}{\zeta} \cdot \log(\omega) \]

\[ 90 + 90 \cdot \log(\omega) \]

This new approximation is a better fit to the phase function $\Phi(\omega)$ for the case $\zeta = 1$ than the approximation whose slope is $\frac{132}{\zeta}$ degrees per decade. To extend the result to the case where $\zeta$ does not equal 1, let us examine the behavior of the phase function as $\zeta$ increases. From the plots of $\Phi(\omega)$ for various values of $\zeta$, we see that as $\zeta$ decreases, the slope of the phase function evaluated at $\omega_0$ increases. To make our new approximation's slope increase as $\zeta$ decreases, we take as the slope of our straight-line approximation $\frac{90}{\zeta}$ degrees per decade. The results are shown below (along with the $\frac{132}{\zeta}$ degree per decade lines) for several values of $\zeta$. 
\[ \zeta := 0.8 \]

\[
\arg \left[ 2 \cdot \zeta \cdot \omega_o \cdot \omega \cdot i + \left( \omega_o^2 - \omega^2 \right) \right] \cdot \frac{360}{2 \cdot \pi} \\
90 + \frac{132}{\zeta} \cdot \log(\omega) \\
90 + \frac{90}{\zeta} \cdot \log(\omega)
\]

\[ \zeta := 0.6 \]

\[
\arg \left[ 2 \cdot \zeta \cdot \omega_o \cdot \omega \cdot i + \left( \omega_o^2 - \omega^2 \right) \right] \cdot \frac{360}{2 \cdot \pi} \\
90 + \frac{132}{\zeta} \cdot \log(\omega) \\
90 + \frac{90}{\zeta} \cdot \log(\omega)
\]
\( \zeta := .4 \)

\[
\arg \left[ 2 \cdot \zeta \cdot \omega_0 \cdot \omega \cdot i + \left( \omega_0^2 - \omega^2 \right) \right] \cdot \frac{360}{2 \cdot \pi} - 90 + \frac{132}{\zeta} \cdot \log(\omega) - 90 + \frac{90}{\zeta} \cdot \log(\omega)
\]

\( \zeta := .2 \)

\[
\arg \left[ 2 \cdot \zeta \cdot \omega_0 \cdot \omega \cdot i + \left( \omega_0^2 - \omega^2 \right) \right] \cdot \frac{360}{2 \cdot \pi} - 90 + \frac{132}{\zeta} \cdot \log(\omega) - 90 + \frac{90}{\zeta} \cdot \log(\omega)
\]
In order to calculate the error at $\omega_{\text{lo}}$ and $\omega_{\text{hi}}$, we must produce formulae for $\omega_{\text{lo}}$ and $\omega_{\text{hi}}$ in terms of $\zeta$. Since

$$180 = 90 + \frac{90}{\zeta} \cdot \log\left( \frac{\omega_{\text{hi}}}{\omega_o} \right)$$

we have $\omega_{\text{hi}} = \omega_o \cdot 10^{\frac{\zeta}{\zeta}}$

Similarly, since $0 = 90 + \frac{90}{\zeta} \cdot \log\left( \frac{\omega_{\text{lo}}}{\omega_o} \right)$ we have $\omega_{\text{lo}} = \omega_o \cdot 10^{-\frac{\zeta}{\zeta}}$

Substituting these values into the phase function gives the error at $\omega_{\text{lo}}$ and $\omega_{\text{hi}}$.

$$\text{Error1}_{\text{lo}} = \text{atan}\left[ 2 \cdot \zeta \cdot \omega_o \cdot \frac{10^{(-\zeta)}}{\left( \omega_o^2 - \omega_o^2 \cdot 10^{(-\zeta)} \right)^2} \right]$$

$$= -\text{atan}\left[ 2 \cdot \zeta \cdot \frac{10^{(-\zeta)}}{-1 + 100^{(-\zeta)}} \right]$$

$\zeta := 0.1, 0.2, 1.0$
Error1_{lo}(\zeta) := \left[ -\arctan \left( 2 \cdot \zeta \cdot \frac{10^{(-\zeta)}}{-1 + 100^{(-\zeta)}} \right) \right] \cdot \frac{360}{2 \cdot \pi}

Error1_{lo}(\zeta) =

\begin{array}{c}
23.291 \\
22.749 \\
21.88 \\
20.73 \\
19.36 \\
17.834 \\
16.221 \\
14.58 \\
12.967 \\
11.421
\end{array}

Error1_{hi}(\zeta) = \frac{\pi}{2} - \left[ \frac{\pi}{2} + \arctan \left( 2 \cdot \zeta \cdot \omega_o^2 \cdot \frac{10^\zeta}{\omega_o^2 - \omega_o^2 \cdot (10^\zeta)^2} \right) \right]

= \arctan \left( 2 \cdot \zeta \cdot \frac{10^\zeta}{(-1 + 100^\zeta)} \right)

Error1_{hi}(\zeta) := \arctan \left( 2 \cdot \zeta \cdot \frac{10^\zeta}{(-1 + 100^\zeta)} \right) \cdot \frac{360}{2 \cdot \pi}

Error1_{hi}(\zeta) =

\begin{array}{c}
23.291 \\
22.749 \\
21.88 \\
20.73 \\
19.36 \\
17.834 \\
16.221 \\
14.58 \\
12.967 \\
11.421
\end{array}
Summary of Second Order Phase Approximations

We see that the error for the \( \frac{90}{\zeta} \) degree per decade line is considerably smaller at both \( \omega_{lo} \) and \( \omega_{hi} \) than the error for the \( \frac{132}{\zeta} \) degree per decade line, demonstrating the superiority of the fit to the \( \frac{90}{\zeta} \) degree per decade line. We have shown a straight-line approximation to the second order phase function that is easy to draw and has easily remembered intercepts. The line passes through \( \omega_0 \) at \( 90^\circ \) and has a slope of \( \frac{90}{\zeta} \) degrees per decade. The intercepts are: \( \omega_{hi} = \omega_0 \cdot 10^\zeta \) and \( \omega_{lo} = \omega_0 \cdot 10^{-\zeta} \).

Example #1: \( H(s) = \frac{s^2 + 2 \cdot s + 25}{25} \cdot \left( \frac{s^2}{25} + \frac{2}{25} \cdot s + 1 \right) \)

The constant term is greater than zero and so contributes nothing to the phase plot.

\[
\omega_0 = 5 \quad \zeta = 0.2 \quad \omega_{hi} = 7.924 \quad \omega_{lo} = 3.155 \quad \text{and the slope is} \quad \frac{90^\circ}{\zeta} = 450^\circ \text{ per decade.}
\]

\( \omega := 0.01, 0.02, \ldots, 1000 \)

\[
\text{phase}(\omega) := \begin{cases} 
\omega > 3.155, & \text{if} \left( \omega < 7.924, 90 + 450 \cdot \log\left( \frac{\omega}{5} \right), 180 \right), 0 
\end{cases}
\]
The situation for poles is analogous as shown below.

Example #2: \( H(s) = \frac{1}{(s^2 + 2 \cdot s + 25)} = \frac{1}{25} \cdot \frac{1}{(\frac{s^2}{25} + 2 \cdot \frac{s}{25} + 1)} \)

The constant term is greater than zero and so contributes nothing to the phase plot.

\[ \omega_0 = 5 \quad \zeta = 0.2 \quad \omega_{hi} = 7.924 \quad \omega_{lo} = 3.155 \quad \text{and the slope is } \frac{90^\circ}{\zeta} = 450^\circ \text{ per decade.} \]

\[ \omega := 0.01, 0.02, \ldots, 1000 \]

\[ \text{phase}(\omega) := \begin{cases} \omega > 3.155, & -90 - 450 \cdot \log \left( \frac{\omega}{5} \right), -180, 0 \\ \omega < 7.924, & \end{cases} \]

![Phase Plot](image-url)
Final Example Magnitude Response

$$H(s) = \frac{\left( s + \frac{1}{5}\right) \cdot \left( s^2 + 4.8 \cdot s + 9\right)}{(s + 80) \cdot \left( s^2 + 8 \cdot s + 400\right)} = \frac{1}{5} \cdot 9 \cdot \frac{\left( \frac{s}{80} + 1\right) \cdot \left( \frac{s^2}{400} + \frac{8}{400} \cdot s + 1\right)}{\left( \frac{s}{15} + 1\right) \cdot \left( \frac{s^2}{9} + \frac{4.8}{9} + 1\right)}$$

$$\omega := 0.01, 0.02, \ldots, 1000$$

$$K_0 := 20 \cdot \log \left( \frac{1}{5} \cdot 9 \cdot \frac{1}{80 \cdot 400} \right) \quad K_0 = -84.998$$

$$M_1(\omega) := \text{if } \begin{cases} \omega < \frac{1}{5}, 0, & 20 \cdot \log \left( \frac{\omega}{1/5} \right) \end{cases}$$

$$M_2(\omega) := \text{if } \begin{cases} \omega < 80, 0, & -20 \cdot \log \left( \frac{\omega}{80} \right) \end{cases}$$

$$M_3(\omega) := \text{if } \begin{cases} \omega < 3, 0, & 40 \cdot \log \left( \frac{\omega}{3} \right) \end{cases} \quad \omega_3 := 3 \quad \zeta_3 := \frac{4.8}{2 \cdot \sqrt{9}} \quad \zeta_3 = 0.8$$

$$M_4(\omega) := \text{if } \begin{cases} \omega < 20, 0, & -40 \cdot \log \left( \frac{\omega}{20} \right) \end{cases} \quad \omega_4 := 20 \quad \zeta_4 := \frac{8}{2 \cdot \sqrt{400}} \quad \zeta_4 = 0.2$$

$$\text{points} := \begin{bmatrix} 0.01 & K_0 \\ 0.2 & K_0 \\ 3 & K_0 + 20 \cdot \log \left( \frac{3}{0.2} \right) \\ 20 & K_0 + 20 \cdot \log \left( \frac{3}{0.2} \right) + 60 \cdot \log \left( \frac{20}{3} \right) \\ 80 & 0 \\ 1000 & 0 \end{bmatrix}$$
\[ M_T(\omega) := K_0 + M_1(\omega) + M_2(\omega) + M_3(\omega) + M_4(\omega) \]

\[ A_{30} := 20 \cdot \log(2 \cdot \zeta 3) \quad A_{32} := -10 \cdot \log[(0.3 \zeta 3)^2 + 0.563] \]

\[
\begin{pmatrix}
0.3 & K_0 + 20 \cdot \log \left( \frac{0.3}{0.2} \right) \\
1.5 & K_0 + 20 \cdot \log \left( \frac{1.5}{0.2} \right) + A_{32} \\
3 & K_0 + 20 \cdot \log \left( \frac{3}{0.2} \right) + A_{30} \\
10 & K_0 + 20 \cdot \log \left( \frac{3}{0.2} \right) + 60 \cdot \log \left( \frac{10}{3} \right)
\end{pmatrix}
\]

\[
\frac{\omega_{40}}{2} = 10
\]

\[
\omega_{4p} := \omega_{40} \sqrt{1 - 2 \cdot \zeta 4^2} \quad \omega_{4p} = 19.183 \quad \omega_{4c} := \sqrt{2} \cdot \omega_{4p} \quad \omega_{4c} = 27.129
\]

\[ A_{40} := -20 \cdot \log(2 \cdot \zeta 4) \quad A_{40} = 7.959 \quad A_{4p} := -10 \cdot \log\left[4 \cdot \zeta 4^2 \cdot (1 - \zeta 4^2)\right] \]

\[ A_{4p} = 8.136 \quad A_{42} := -10 \cdot \log\left(\zeta 4^2 + 0.5625\right) \quad A_{42} = 2.2
\]

\[
\begin{pmatrix}
\frac{\omega_{40}}{2} & K_0 + 20 \cdot \log \left( \frac{3}{0.2} \right) + 60 \cdot \log \left( \frac{10}{3} \right) + A_{42} \\
\omega_{40} & K_0 + 20 \cdot \log \left( \frac{3}{0.2} \right) + 60 \cdot \log \left( \frac{20}{3} \right) + A_{40} \\
\omega_{4c} & K_0 + 20 \cdot \log \left( \frac{3}{0.2} \right) + 60 \cdot \log \left( \frac{20}{3} \right) + 20 \cdot \log \left( \frac{\omega_{4c}}{20} \right)
\end{pmatrix}
\]
Final Example Phase Response

points :=

\[
\begin{pmatrix}
0.01 & 0 \\
0.2 & 0 \\
0.2 & 90 \\
3 & 90 \\
3 & 270 \\
20 & 270 \\
20 & 90 \\
80 & 90 \\
80 & 0 \\
1000 & 0
\end{pmatrix}
\]

\[
A := \begin{pmatrix}
0.02 & 0 \\
2 & 90
\end{pmatrix}
\]

\[
\omega_{4_{\text{lo}}} := \omega_{4_{0}} \cdot 10^{-\zeta^4}
\]

\[
\omega_{4_{\text{hi}}} := \omega_{4_{0}} \cdot 10^{\zeta^4}
\]

\[
B := \begin{pmatrix}
\omega_{3_{\text{lo}}} & 90 \\
\omega_{3_{\text{hi}}} & 270
\end{pmatrix}
\]

\[
\omega_{3_{\text{lo}}} := \omega_{3_{0}} \cdot 10^{-\zeta^3}
\]

\[
\omega_{3_{\text{hi}}} := \omega_{3_{0}} \cdot 10^{\zeta^3}
\]

\[
C := \begin{pmatrix}
\omega_{4_{\text{lo}}} & 270 \\
\omega_{4_{\text{hi}}} & 90
\end{pmatrix}
\]

\[
D := \begin{pmatrix}
8 & 90 \\
800 & 0
\end{pmatrix}
\]

\[
\frac{90}{\zeta^3} = 112.5
\]

\[
\frac{90}{\zeta^4} = 450
\]
For $\omega < 0.02$ \text{ slope } = 0, \text{ for } 0.02 < \omega < \omega_{3\text{lo}} \text{ slope } = +45, \text{ for } \omega_{3\text{lo}} < \omega < 2 \text{ slope } = +45 + 112.5 = 157.5, \text{ for } 2 < \omega < 8 \text{ slope } = +112.5, \text{ for } 8 < \omega < \omega_{4\text{lo}} \text{ slope } = +112.5 - 45 = 67.5, \text{ for } \omega_{4\text{lo}} < \omega < \omega_{3\text{hi}} \text{ slope } = +112.5 - 450 - 45 = -382.5, \text{ for } \omega_{3\text{hi}} < \omega < \omega_{4\text{hi}} \text{ slope } = -450 - 45 = -495, \text{ for } \omega_{4\text{hi}} < \omega < 800 \text{ slope } = -45

Point #1 = (0.01,0), point #2 = (0.02,0). \quad 0 + 45 \cdot \log \left( \frac{\omega_{3\text{lo}}}{0.02} \right) = 61.924 \text{ so point } #3,

61.925 + 157.5 \cdot \log \left( \frac{2}{\omega_{3\text{lo}}} \right) = 160.191 \text{ so point } #4.
\[ 160.191 + 112.5 \cdot \log \left( \frac{8}{2} \right) = 227.923 \text{ so point #5,} \]

\[ 227.923 + 67.5 \cdot \log \left( \frac{\omega_{4lo}}{8} \right) = 241.284 \text{ so point #6.} \]

\[ 241.284 - 382.5 \cdot \log \left( \frac{\omega_{3hi}}{\omega_{4lo}} \right) = 173.929 \text{ so point #7,} \]

\[ 173.929 - 495 \cdot \log \left( \frac{\omega_{4hi}}{\omega_{3hi}} \right) = 63.094 \text{ so point #8} \]

\[ 63.094 - 45 \cdot \log \left( \frac{800}{\omega_{4hi}} \right) = 0 \text{ so point #9 = (800,0) and point #10 = (1000,0).} \]

\[
\begin{pmatrix}
0.01 & 0 \\
0.02 & 0 \\
\omega_{3lo} & 61.924 \\
2 & 160.191 \\
8 & 227.933 \\
\omega_{4lo} & 241.284 \\
\omega_{3hi} & 173.929 \\
\omega_{4hi} & 63.094 \\
800 & 0 \\
1000 & 0
\end{pmatrix}
\]

points :=
\[
\Phi_1(\omega) := \begin{cases} 
\omega < 0.02, & 0 \\
\omega < 2, & 45 + 45 \cdot \log\left(\frac{\omega}{0.2}\right), 90 
\end{cases}
\]

\[
\Phi_2(\omega) := \begin{cases} 
\omega < 8, & 0 \\
\omega < 800, & -45 - 45 \cdot \log\left(\frac{\omega}{80}\right), -90 
\end{cases}
\]

\[
\Phi_3(\omega) := \begin{cases} 
\omega < \omega_{3\text{lo}}, & 0 \\
\omega < \omega_{3\text{hi}}, & 90 + 112.5 \cdot \log\left(\frac{\omega}{\omega_{3\text{lo}}}\right), 180 
\end{cases}
\]

\[
\Phi_4(\omega) := \begin{cases} 
\omega < \omega_{4\text{lo}}, & 0 \\
\omega < \omega_{4\text{hi}}, & -90 - 450 \cdot \log\left(\frac{\omega}{\omega_{4\text{lo}}}\right), -180 
\end{cases}
\]

\[
\Phi_T(\omega) := \Phi_1(\omega) + \Phi_2(\omega) + \Phi_3(\omega) + \Phi_4(\omega)
\]
References